A large deviation principle for networks of rate neurons with correlated synaptic weights

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Abstract

We study the asymptotic law of a network of interacting neurons when the number of neurons becomes infinite. Given a completely connected network of firing rate neurons in which the synaptic weights are Gaussian *c*orrelated random variables, we describe the asymptotic law of the network when the number of neurons goes to infinity. We introduce the process-level empirical measure of the trajectories of the solutions to the equations of the finite network of neurons and the averaged law (with respect to the synaptic weights) of the trajectories of the solutions to the equations of the network of neurons.

The main result of this article is that the image law through the empirical measure satisfies a large deviation principle with a good rate function which is shown to have a unique global minimum. Our analysis of the rate function allows us also to characterize the limit measure as the image of a stationary Gaussian measure defined on a transformed set of trajectories. This is potentially very useful for applications in neuroscience since the Gaussian measure can be completely characterized by its mean and spectral density. It also facilitates the assessment of the probability of finite-size effects.

1 Introduction

The goal of this paper is to study the asymptotic behaviour and large deviations of a network of interacting neurons when the number of neurons becomes infinite. Our network may be thought of as a network of weakly-interacting diffusions: thus before we begin we briefly overview other asymptotic analyses of such systems. In particular, a lot of work has been done on spin glass dynamics, including Ben Arous and Guionnet on the mathematical side [29, 3, 4, 30] and Sompolinsky and his co-workers on the theoretical physics side [39, 40, 11, 12]. Furthermore the large deviations of weakly interacting diffusions has been extensively studied by Dawson, Gartner and co-workers [16, 17, 15]. More references to previous work on this particular subject can be found in these references.

Because the dynamics of spin glasses is not too far from that of networks of interacting neurons, Sompolinsky also succesfully explored this particular topic [38] for fully connected networks of rate neurons, i.e. neurons represented by the time variation of their firing rates (the number of spikes they emit per unit of time), as opposed to spiking neurons, i.e. neurons represented by the time variation of their membrane potential (including the individual spikes). For an introduction to these notions, the interested reader is referred to such textbooks as [26, 31, 24]. In his study of the continuous time dynamics of networks of rate neurons, Sompolinsky and his colleagues assumed, as in the work on spin glasses, that the coupling coefficients, called the synaptic weights in neuroscience, were random variables i.i.d. with zero mean Gaussian laws. The main result obtained by Ben Arous and Guionnet for spin glass networks using a large deviations approach (resp. by Sompolinsky and his colleagues for networks of rate neurons using the local chaos hypothesis) under the previous hypotheses is that the averaged law of Langevin spin glass

(resp. rate neurons) dynamics is chaotic in the sense that the averaged law of a finite number of spins (resp. of neurons) converges to a product measure.

The next theoretical efforts in the direction of understanding the averaged law of rate neurons are those of Cessac, Moynot and Samuelides [9, 34, 35, 10, 37]. From the technical viewpoint, the study of the collective dynamics is done in discrete time, assuming no leak (this term is explained below) in the individual dynamics of each of the rate neurons. Moynot and Samuelides obtained a large deviation principle and were able to describe in detail the limit averaged law that had been obtained by Cessac using the local chaos hypothesis and to prove rigourously the propagation of chaos property. Moynot extended these results to the more general case where the neurons can belong to two populations, the synaptic weights are non-Gaussian (with some restrictions) but still i.i.d., and the network is not fully connected (with some restrictions) [34].

One of the next challenges is to incorporate in the network model the fact that the synaptic weights are not independent and in effect often highly correlated. One of the reasons for this is the plasticity processes at work at the levels of the synaptic connections between neurons; see for example [32] for a biological viewpoint, and [18, 26, 24] for a more computational and mathematical account of these phenomena.

The problem we solve in this paper is the following. Given a completely connected network of firing rate neurons in which the synaptic weights are Gaussian *c*orrelated random variables, we describe the asymptotic law of the network when the number of neurons goes to infinity. Like in [34, 35] we study a discrete time dynamics but unlike these authors we cope with more complex intrinsic dynamics of the neurons, in particular we allow for a leak (to be explained in more detail below). The structure of our proof is broadly similar to these authors; we have generalised their results. Indeed one may directly obtain the LDP in [34] by applying a contraction principle to the LDP to be proved below.

To be complete, let us mention the fact that this problem has already partially been explored in Physics by Sompolinsky and Zippelius [39, 40] and in Mathematics by Alice Guionnet [30] who analysed symmetric spin glass dynamics, i.e. the case where the matrix of the coupling coefficients (the synaptic weights in our case) is symmetric. This is a very special case of correlation. The work in [13] is also an important step forward in the direction of understanding the spin glass dynamics when more general correlations are present.

Let us also mention very briefly another class of approaches toward the description of very large populations of neurons where the individual spikes generated by the neurons are considered. The model for individual neurons is usually of the class of Integrate and Fire (IF) neurons [33] and the underlying mathematical tools are those of the theory of point-processes [14]. Important results have been obtained in this framework by Gerstner and his collaborators, e.g. [27, 25] in the case of deterministic synaptic weights. Related to this approach but from a more mathematical viewpoint, important results on the solutions of the mean-field equations have been obtained in [8]. In the case of spiking neurons but with a continuous dynamics (unlike that of IF neurons), the first author and collaborators have recently obtained some limit equations that describe the asymptotic dynamics of fully connected networks of neurons [1] with independent synaptic weights.

Because of the correlation of the synaptic weights, the natural space to work in is the infinite dimensional space of the trajectories, noted $\mathcal{T}^{\mathbb{Z}}$, of a countably-infinite set of neurons and the set of stationary probability measures defined on this set, noted $\mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$.

We introduce the process-level empirical measure, noted $\hat{\mu}^N$, of the N trajectories of the solutions to the equations of the network of N neurons and the averaged (with respect to the synaptic weights) law Q^N of the N trajectories of the solutions to the equations of the network of N neurons. The main result of this article (theorem 2) is that the image law Π^N of Q^N through μ^N satisfies a large deviation principle (LDP) with a good rate function H which is shown to have a unique global minimum, μ_e. Thus, with respect to the measure Π^N on $\mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$, if the set X contains the measure δ_{μ_e}, then $\Pi^N(X) \to 1$ as $N \to \infty$, whereas if δ_{μ_e} is not in the closure of X, $\Pi^N(X) \to 0$ as $N \to \infty$ exponentially fast and the constant in the exponential rate is determined by the rate function. Our analysis of the rate function allows us also to characterize the limit measure μ_e as the image of a stationary Gaussian measure $\underline{\underline{\mu_e}}$ defined on a transformed set of trajectories $\mathcal{S}^{\mathbb{Z}}$. This is potentially very useful for applications since $\underline{\underline{\mu_e}}$ can be completely characterized by its mean and spectral density. Furthermore the rate function allows us to quantify the probability of finite-size effects.

The paper is organized as follows. In section 2 we describe the equations of our network of neurons, the type of correlation between the synaptic weights, define the proper state spaces and introduce the different probability measures that are necessary for establishing our results, in particular

the level-3 empirical measure, $\hat{\mu}^N$, Π^N and the image R^N through $\hat{\mu}^N$ of the law of the uncoupled neurons. We state the principle result of this paper in Theorem 2. In section 3 we motivate our approach by showing that when computing the Radon-Nikodym derivative of Q^N with respect to the law of the uncoupled neurons, one is led to consider certain Gaussian processes which are directly related to the synaptic weights and can be described with the help of the empirical measure $\hat{\mu}^N$.

In section 4 we extend the definition of the previous Gaussian processes to be valid for any stationary measure, not only the empirical one. This allows us to compute the Radon-Nikodym derivative of Π^N with respect to R^N for any measure in $\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$. Using these results, section 5 is dedicated to the proof of the existence of a strong LDP for the measure Π^N. In section 6 we show that the good rate function obtained in the previous section has a unique global minimum and we characterize it as the image of a stationary Gaussian measure. We conclude with section 7 by stating some important consequences and sketching a number of possible generalisations of our work as well as discussing some further connections with other approaches.

2 The neural network model

We consider a fully connected network of N rate neurons. For simplicity but without loss of generality, we assume N odd[1] and write $N = 2n+1$, $n \geq 0$. In the course of this paper, we will asymptote N to ∞, so that unless otherwise stated the parameters are taken to be independent of N. The state of the neurons is described either by the rate variables (X_t^j), $j = -n, \cdots, n$, $t = 0, \cdots, T$ or the potential variables (U_t^j), $j = -n, \cdots, n$, $t = 0, \cdots, T$. These variables are related as follows

$$X_t^j = f(U_t^j) \quad j = -n, \ldots, n \quad t = 0, \ldots, T-1,$$

where $f : \mathbb{R} \to]0,\, 1[$ is a monotonic bijection. We could for example employ $f(x) = (1 + \tanh(gx))/2$, where the parameter g can be used to control the slope of the "sigmoid" f at the origin $x = 0$.

We consider the case where the time variable t takes the $T+1$ discrete integer values $0, 1, \ldots, T$ because it simplifies the problem. We leave for future work the case of the continuous time variable.

[1]When N is even the formulae are slightly more complicated but all the results we prove below in the case N odd are still valid.

2.1 The equations

The equation describing the time variation of the membrane potential U^j of the jth neuron writes

$$U_t^j = \gamma U_{t-1}^j + \sum_{i=-n}^{n} J_{ji} f(U_{t-1}^i) + \theta_j + B_{t-1}^j, \quad j = -n, \dots, n \quad t = 1, \dots, T. \tag{1}$$

This equation involves the parameters γ, J_{ij}, θ_j, and B_t^j, $i, j = -n, \dots, n$, $t = 0, \dots, T-1$.

γ is a positive real between 0 and 1 that determines the time scale of the intrinsic dynamics, i.e. without interactions, of the neurons. If $\gamma = 0$ the dynamics is said to have no leak.

The J_{ij}s are the synaptic weights. J_{ij} represents the strength with which the 'presynaptic' neuron j influences the 'postsynaptic' neuron i. They are random *variables* whose laws are described below.

The θ_js are the thresholds: they change the value of the potential of the neuron j at which the sigmoid f takes the value 1/2. Like the J_{ij}s they are random *variables* that we assume to be i.i.d. as $\mathcal{N}_1(\bar{\theta}, \theta^2)$, and independent of the J_{ij}s[2].

Finally the B_t^js represent random fluctuations of the membrane potential of neuron j. They are independent random *processes* with the same law. We assume that at each time instant t, the B_t^js are i.i.d. random variables distributed as $\mathcal{N}_1(0, \sigma^2)$. They are also independent of the synaptic weights and the thresholds.

The equation corresponding to (1) for the rates writes

$$X_t^j = f\left(\gamma f^{-1}(X_{t-1}^j) + \sum_{i=-n}^{n} J_{ji} X_{t-1}^i + \theta_j + B_{t-1}^j\right), \tag{2}$$

where $j = -n, \cdots, n \quad t = 1, \cdots, T.$. Note that the values of the rates are in the open interval $]0, 1[$.

[2]We note $\mathcal{N}_p(m, \Sigma)$ the law of the p-dimensional Gaussian variable with mean m and covariance matrix Σ.

2.2 The law of the synaptic weights

The N^2 synaptic weights are modelled as Gaussian random variables. We assume that they have the same mean which scales as $1/N$:

$$\mathbb{E}\left[J_{ij}\right] = \frac{\bar{J}}{N} \quad i,\, j = -n, \cdots, n. \tag{3}$$

We next specify their covariance structure. The covariance is assumed to satisfy the following symmetry,

$$cov(J_{ij}J_{kl}) = cov(J_{i+m,j+n}J_{k+m,l+n})$$

for all indexes $i,\, j,\, k,\, l = -n, \cdots, n$ and all integers m and n, the indexes being taken modulo N. We may interpret this property by imagining that the neurons are arranged on a ring with the following 'shift invariance'. If we fix two presynaptic neurons and shift the postsynaptic neurons, then the correlations are invariant. Similarly if we fix two postsynaptic neurons and shift the presynaptic neurons, the correlations are invariant.

We stipulate the covariance through a function $\Lambda : \mathbb{Z}^2 \to \mathbb{R}$, which satisfies

$$\Lambda(\varepsilon_1 k, \varepsilon_2 l) = \Lambda(k,l) \;\; \varepsilon_1,\, \varepsilon_2 = \pm 1. \tag{4}$$

We assume furthermore that the covariances scale as $1/N$. We write

$$cov(J_{ij}J_{kl}) = \frac{1}{N}\Lambda\left((i-k) \bmod N, (j-l) \bmod N\right). \tag{5}$$

Here, and throughout this paper, $i \bmod N$ is taken to lie between $-n$ and n. It is important to note that the covariance function Λ and mean $\bar{J}$ are independent of N, so that these remain fixed when we asymptote N to infinity later on. We let Λ^N be the restriction of Λ to $[-n,n]^2$, i.e. $\Lambda^N(i,j) = \Lambda(i,j)$ for $-n \le i, j \le n$.

Being a covariance function (up to the scale factor $1/N$) $\Lambda^N(i,j)$ must be a positive-definite function, i.e. it satisfies

$$\sum_{r,s,k,l=-n}^{n} \Lambda^N(r-k, s-l)\lambda_{rs}\lambda_{kl} \ge 0,$$

for all reals λ_{rs}, λ_{kl}, the indexing being taken modulo N.

The fact that Λ^N is a positive-definite function imposes that its two dimensional discrete Fourier transform (DFT), noted $\tilde{\Lambda}^N$, is positive[3], and also called its spectral density or power spectrum. In detail

$$\tilde{\Lambda}^N(p,q) = \sum_{k,\,l=-n}^{n} \Lambda(k,l) e^{-\frac{2\pi i}{N}(pk+ql)} \quad p,\, q = -n, \cdots, n. \tag{6}$$

Conversely, the values of the covariances can be recovered from the Inverse DFT of the sequence $(\tilde{\Lambda}^N(p,q))_{p,\,q=-n,\cdots,n}$, i.e.

$$\Lambda(k,l) = \frac{1}{N^2} \sum_{k,\,l=-n}^{n} \tilde{\Lambda}^N(p,q) e^{\frac{2\pi i}{N}(pk+ql)} \quad k,\, l = -n, \cdots, n.$$

We must make further assumptions on Λ to ensure that the system is well-behaved as the number of neurons N asymptotes to infinity. We assume that the series $(\Lambda(k,l))_{k,\,l\in\mathbb{Z}})$ is absolutely convergent, i.e.

$$\Lambda^{sum} = \sum_{k,l=-\infty}^{\infty} |\Lambda(k,l)| < \infty. \tag{7}$$

In practice one might expect there to exist a maximal correlation distance d such that $\Lambda(k,l) = 0$ if $|k| + |l| > d$ (especially since in practice there is only a finite number of neurons). The existence of such a maximal correlation distance would be sufficient to guarantee the requirement (7), however we refrain from explicitly making this assumption as it is not necessary *per se*. It follows from (7) that the series $\sum_{k,l=-\infty}^{\infty} \Lambda(k,l)e^{-i(k\omega_1+l\omega_2)}$ is absolutely convergent and defines a continuous function, noted $\tilde{\Lambda}(\omega_1,\omega_2)$, on $[-\pi,\pi[^2$ such that:

$$\Lambda(k,l) = \frac{1}{(2\pi)^2} \int_{[-\pi,\,\pi[^2} e^{i(k\omega_1+l\omega_2)}\, \tilde{\Lambda}(\omega_1,\omega_2)\, d\omega_1\, d\omega_2 \quad k,\, l \in \mathbb{Z}.$$

The continuous function $\tilde{\Lambda}$ is termed the spectral density of $(\Lambda(k,l))_{k,l\in\mathbb{Z}}$. It is clear from the definitions that, if the series (p_N) and (q_N) satisfy $\lim_{N\to\infty} 2\pi p_N/N = \omega_1$ and $\lim_{N\to\infty} 2\pi q_N/N = \omega_2$, then

$$\tilde{\Lambda}^N(p_N, q_N) \to \tilde{\Lambda}(\omega_1,\omega_2).$$

[3]This is a standard result in Fourier Analysis, see lemma 5.

Since Λ^N is a positive-definite function (for all N),

$$\tilde{\Lambda}(\omega_1, \omega_2) \geq 0 \quad \forall(\omega_1, \omega_2) \in [-\pi, \pi[^2.$$

We also assume that there exists $\tilde{\Lambda}^{min}$ such that, for all N

$$\tilde{\Lambda}^N(0,0) \geq \tilde{\Lambda}^{\min} > 0. \tag{8}$$

2.3 The laws of the uncoupled and coupled processes

The trajectories of the $(X^j)_{t=0\cdots T}$ defined by (2) are points in the $T+1$-dimensional space $]0,1[^{[0\cdots T]} \stackrel{\text{def}}{\equiv} \mathcal{T}$. The law of the solution of (2) is a probability measure on $\mathcal{T}^N$. We note $\mathcal{M}_1^+(\mathcal{T}^N)$ the set of probability measures on $\mathcal{T}^N$.

The trajectories of the $(U^j)_{t=0,T}$ defined by (1) are points in $\mathbb{R}^{[0,T]} \stackrel{\text{def}}{\equiv} \mathcal{S}$. The law of the solution of (1) is a probability measure on $\mathcal{S}^N$. This law is the image of the probability measure of that of the $(X^j)_{t=0\cdots T}$ by the function f. We note $\mathcal{M}_1^+(\mathcal{S}^N)$ the set of probability measures on $\mathcal{S}^N$.

2.3.1 The uncoupled processes and the initial conditions

We specify the initial conditions for (2) as N i.i.d. random variables $(X_0^j)_{j=-n,\cdots,n}$. Let μ_I be the individual law on the interval $]0,1[$ of X^j; it follows that the joint law of the variables is $\mu_I^{\otimes N}$ on $]0,1[^N$. The initial conditions for (1), noted $(U_0^j)_{j=-n,\cdots,n}$, are also i.i.d. with law $\underline{\mu}_I^{\otimes N}$, where $\underline{\mu}_I \stackrel{\text{def}}{\equiv} \mu_I \circ f$ is the image on $\mathbb{R}$ of the law μ_I defined on $]0,1[$ through the function f. Throughout this paper we employ the convention that if $x \in \mathcal{T}$ then $f(x) = (f(x_0), \dots, f(x_T))$, and if $x = (x^{-n}, \dots, x^n) \in \mathcal{T}^N$ then $f(x) = (f(x^{-n}), \dots, f(x^n))$. We assume that $\underline{\mu}_I$ is Gaussian under a change of variable, i.e. there exists Ψ_0, a continuous bijection on $\mathbb{R}$, such that

$$\underline{\mu}_I = \mathcal{N}_1(0, \sigma^2) \circ \Psi_0, \tag{9}$$

We note P the law of the solution to one of the uncoupled equations (2) where we take θ_j deterministic and equal to $\bar{\theta}$ and $J_{ij} = 0$, i, $j = -n, \cdots, n$. P is the law of the solution to the following stochastic difference equation:

$$X_t = f\left(\gamma f^{-1}(X_{t-1}) + \bar{\theta} + B_{t-1}\right), \; t = 1, \cdots, T$$

with the law of the initial condition being μ_I. Hence the image $\underline{P}$ of P through f is the law of the solution to the following stochastic difference equation

$$U_t = \gamma U_{t-1} + \bar{\theta} + B_{t-1},\ t = 1, \cdots, T \tag{10}$$

the law of the initial condition being $\underline{\mu}_I$. This last process can be characterized exactly, as follows.

Let $\Psi : \mathcal{S} \to \mathcal{S}$ be the continuous bijection

$$\Psi(u) = {}^t(v_0, v_1, \ldots, v_T), \tag{11}$$

where $v_0 = \Psi_0(u_0)$ and for $1 \leq s \leq T$,

$$v_s = \Psi_s(u) = u_s - \gamma u_{s-1} - \bar{\theta} \quad s = 1, \cdots, T. \tag{12}$$

We employ the convention that if $u = (u^{-n}, \ldots, u^n) \in \mathcal{S}^N$ then $\Psi(u) = (\Psi(u^{-n}), \ldots, \Psi(u^n))$. The following proposition is evident from equations (9), (10) and (12).

Proposition 1. *The law $\underline{P}$ of the solution to* (10) *writes*

$$\underline{P} = \mathcal{N}_{T+1}(\mathbf{0}_{T+1}, \sigma^2 \mathrm{Id}_{T+1}) \circ \Psi,$$

where $\mathbf{0}_{T+1}$ is the $T+1$-dimensional vector of coordinates equal to 0 and Id_{T+1} is the $T+1$-dimensional identity matrix.

2.3.2 Coupled processes

If we reintroduce the coupling between the neurons, we note $Q^N(J, \theta)$ the conditional law of the (X_t^j), $j = -n, \cdots, n$, $t = 0, \cdots, T$, solution to (2), for given (J, Θ). It is an element of $\mathcal{M}_1^+(\mathcal{T}^N)$. Similarly $\underline{Q}^N(J, \theta)$ is the corresponding law of the (U_t^j), $j = -n, \cdots, n$, $t = 0, \cdots, T$, solution to (1), for given (J, Θ). We let $Q^N = \mathbb{E}^{J,\Theta}[Q^N(J, \Theta)]$ and $\underline{Q}^N = \mathbb{E}^{J,\Theta}[\underline{Q}^N(J, \Theta)]$ be the laws averaged with respect to the weights and thresholds.

2.3.3 Infinite number of neurons

We note $\mathcal{T}^{\mathbb{Z}}$ the set of doubly infinite elements of $\mathcal{T}$. If $x = (x^i)_{i=-\infty,\ldots,\infty}$ is in $\mathcal{T}^{\mathbb{Z}}$, we note x^i, $i \in \mathbb{Z}$ its ith coordinate. We define the projection

$\pi_N : \mathcal{T}^{\mathbb{Z}} \to \mathcal{T}^N$ ($N = 2n+1$) to be $\pi_N(x) = (x^{-n}, \ldots, x^n)$. The shift operator $S : \mathcal{T}^{\mathbb{Z}} \to \mathcal{T}^{\mathbb{Z}}$ is defined by

$$(Sx)^i = x^{i+1}, \quad i \in \mathbb{Z}$$

Given the element $(x^{-n}, \ldots, x^n)$ of $\mathcal{T}^N$ we form the doubly infinite periodic sequence

$$x(N) = (\ldots, x^{n-1}, x^n, x^{-n}, \ldots, x^n, x^{-n}, x^{-n+1}, \ldots)$$

which is an element of $\mathcal{T}^{\mathbb{Z}}$. We have $(x(N))^i = x^{(i \mod N)}$, where $i \mod N$ lies between $-n$ and n.

We equip $\mathcal{T}^{\mathbb{Z}}$ with the projective topology, i.e. the topology generated by the following metric. For $x, y \in \mathcal{T}^N$, let

$$d_N(x, y) = \sup_{|j| \leq n, 0 \leq s \leq T} \left| x_s^j - y_s^j \right|.$$

This allows us to define the following metric over $\mathcal{T}^{\mathbb{Z}}$, whereby if $x, y \in \mathcal{T}^{\mathbb{Z}}$, then

$$d(x, y) = \sum_{N=1}^{\infty} 2^{-N} d_N(\pi_N x, \pi_N y). \tag{13}$$

The metrics d_N and d generate, respectively, the Borelian sigma-algebras $\mathcal{B}(\mathcal{T}^N)$ and $\mathcal{B}(\mathcal{T}^{\mathbb{Z}})$. We note that $\mathcal{T}^Z$ is Polish (a complete, separable metric space).

A strictly stationary measure μ on $\mathcal{T}^{\mathbb{Z}}$ with its Borelian sigma-algebra satisfies

$$\mu(S(B)) = \mu(B) \quad \forall B \in \mathcal{B}(\mathcal{T}^{\mathbb{Z}}).$$

Its N-dimensional marginal[4] also satisfies

$$\mu^N(S(B)) = \mu^N(B) \quad \forall B \in \mathcal{B}(\mathcal{T}^N)$$

where for each $x \in \mathcal{T}^N$ we have defined

$$Sx = \pi_N(S(x(N))).$$

We note $\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$ (respectively $\mathcal{M}_{1,s}^+(\mathcal{T}^N)$) the set of strictly stationary probability measures on $\mathcal{T}^{\mathbb{Z}}$ (respectively on $\mathcal{T}^N$). Note that everything

[4]In what follows, the N-dimensional marginal μ^N of a measure μ in $\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$ is such that $\mu^N = \mu \circ \pi_N^{-1}$.

above applies mutatis mutandis to $\mathcal{S}^Z$ (respectively $\mathcal{S}^N$) and $\mathcal{M}_{1,s}^+(\mathcal{S}^Z)$ (respectively $\mathcal{M}_{1,s}^+(\mathcal{S}^N)$).

We now introduce the following empirical measure. Given an N-tuple $(x^{-n}, \ldots, x^n)$ in $\mathcal{T}^N$ we associate with it the measure, noted $\hat{\mu}^N(x^{-n}, \ldots, x^n)$, in $\mathcal{M}_1^+(\mathcal{T}^{\mathbb{Z}})$ defined by

$$\hat{\mu}^N : \mathcal{T}^N \to \mathcal{M}_1^+(\mathcal{T}^{\mathbb{Z}}) \quad \text{such that} \quad d\hat{\mu}^N(x^{-n}, \cdots, x^n)(y) = \frac{1}{N} \sum_{i=-n}^{n} \delta_{S^i x(N)}(y). \tag{14}$$

We also sometimes consider $\hat{\mu}^N$ to be a function on $\mathcal{T}^{\mathbb{Z}}$ through the projection $\pi_N : \mathcal{T}^{\mathbb{Z}} \to \mathcal{T}^N$.

Note that $\hat{\mu}^N(x^{-n}, \ldots, x^n)$ is a strictly stationary measure on $\mathcal{T}^{\mathbb{Z}}$.

We introduce the following definition and notation.

Definition 1. For each measure $\mu \in \mathcal{M}_{1,s}^+(\mathcal{T}^N)$ or $\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$ we define $\underline{\mu}$ to be $\mu \circ f$.

We next equip $\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$ with the topology of weak convergence, as follows. This can be defined in many ways, but the following definition is the most convenient for our paper. For $\mu^N, \nu^N \in \mathcal{M}_{1,s}^+(\mathcal{T}^N)$, we note the Wasserstein distance

$$d_N(\mu^N, \nu^N) = \inf_{\mathcal{L} \in \mathcal{J}} \left\{ E^{\mathcal{L}}(d_N(x, y)) \right\}, \tag{15}$$

where $\mathcal{J}$ is the set of all measures in $\mathcal{M}_1^+(\mathcal{T}^{2N})$ with N-dimensional marginals μ^N and ν^N. For $\mu, \nu \in \mathcal{M}_{1,s}^+(\mathcal{T}^Z)$, we define

$$d(\mu, \nu) = 2 \sum_{n=0}^{\infty} \kappa_n d_N(\mu^N, \nu^N), \tag{16}$$

where $N = 2n + 1$. Here $\kappa_n = \max(\lambda_n, 2^{-N})$ and $\lambda_n = \sum_{k=-\infty}^{\infty} |\Lambda(k, n)|$. We note that this metric is well-defined because $d_N(\mu^N, \nu^N) \leq 1$ and $\sum_{n=0}^{\infty} \kappa_n < \infty$. It can be shown that $\mathcal{M}_1^+(\mathcal{T}^{\mathbb{Z}})$ is Polish. The topology corresponding to this metric generates a Borelian sigma-algebra which we denote by $\mathcal{B}(\mathcal{M}_1^+(\mathcal{T}^{\mathbb{Z}}))$. The Borelian sigma-algebra on the set of stationary probability measures is denoted by $\mathcal{B}(\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}}))$.

The construction of the topology of $\mathcal{M}_{1,s}^+(\mathcal{S}^N)$ is analogous, except that in (13), for $u, v \in \mathcal{S}^N$, we must replace $d_N(u, v)$ by

$$d_N(u, v)/(1 + d_N(u, v)),$$

the extra division being necessary because the finite-dimensional metric is unbounded. An analogous substitution must also be made in (15).

Using the notation in definition 1 we note $\underline{\hat{\mu}}^N$ the image of $\hat{\mu}^N$ through f. Hence, if $u^i = f^{-1}(x^i)$, $i = -n, \cdots, n$ we have

$$d\underline{\hat{\mu}}^N(u^{-n}, \cdots, u^n)(v) = \frac{1}{N} \sum_{i=-n}^{n} \delta_{S^i u(N)}(v),$$

where the shift operator is defined analogously on $\mathcal{S}^Z$. Since the application Ψ defined in (11) and (12) plays a central role in the sequel we introduce the following definition

Definition 2. For each measure $\underline{\mu} \in \mathcal{M}^+_{1,s}(\mathcal{S}^N)$ or $\mathcal{M}^+_{1,s}(\mathcal{S}^{\mathbb{Z}})$ we define $\underline{\underline{\mu}}$ to be $\underline{\mu} \circ \Psi^{-1}$.

Finally we introduce the image laws in terms of which the principal results of this paper are formulated.

Definition 3.

1. Let Π^N be the image law of Q^N through the function $\hat{\mu}^N : \mathcal{T}^N \to \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ defined by (14).
2. We similarly define R^N to be the image law of $P^{\otimes N}$ under $\hat{\mu}^N$.

That is, $\forall B \in \mathcal{B}(\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}}))$,

$$\Pi^N(B) = Q^N(\hat{\mu}^N \in B) \quad \text{and} \quad R^N(B) = P^{\otimes N}(\hat{\mu}^N \in B).$$

The principal result of this paper is in the next theorem.

Theorem 2. Π^N *is governed by a large deviation principle with a good rate function H (to be defined in definition 6). That is, if F is a closed set in $\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$, then*

$$\overline{\lim_{N\to\infty}} N^{-1} \log \Pi^N(F) \le - \inf_{\mu \in F} H(\mu). \tag{17}$$

Conversely, for all open sets O in $\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$,

$$\underline{\lim_{N\to\infty}} N^{-1} \log \Pi^N(O) \ge - \inf_{\mu \in O} H(\mu). \tag{18}$$

By 'good rate function', we mean that H is not identically ∞ and the sub-level sets

$$\{\mu \in \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}}) : H(\mu) \le c\},$$

where $c \ge 0$, are compact.

3 The Radon-Nikodym derivative of the averaged law of the coupled neurons with respect to the synaptic weights and the thresholds

In the sections to follow we will obtain an LDP for the process with correlations (Q^N) via the (simpler) process without correlations ($P^{\otimes N}$). However in order for us to do this, we must first compute the Radon-Nikodym derivative of Q^N with respect to $P^{\otimes N}$. It is easier to compute the Radon-Nikodym derivative of $\underline{Q}^N$ with respect to $\underline{P}^{\otimes N}$. We do this in the next proposition where, and we will use the same notation throughout the paper, the usual inner product of two vectors u and v of $\mathbb{R}^{T+1}$ is noted $\langle u, v\rangle$.

Proposition 3. *The Radon-Nikodym derivative of $\underline{Q}^N$ with respect to $\underline{P}^{\otimes N}$ is given by the following expression.*

$$\frac{d\underline{Q}^N}{d\underline{P}^{\otimes N}}(u^{-n}, \cdots, u^n) = \mathbb{E}\left[\exp\left(\frac{1}{\sigma^2}\left(\sum_{j=-n}^{n}\langle \Psi(u^j), G^j\rangle - \frac{1}{2}\|G^j\|^2\right)\right)\right], \tag{19}$$

the expectation being taken against the N $(T+1)$-dimensional Gaussian processes (G^i), $i = -n, \cdots, n$ given by

$$\left\{\begin{array}{lcl} G_0^i & = & 0 \\ G_t^i & = & \sum_{j=-n}^{n} J_{ij} f(u_{t-1}^j) + \theta_i - \bar{\theta}, \quad t = 1, \cdots, T, \end{array}\right. \tag{20}$$

and the function Ψ being defined by (11) *and* (12).

Proof. The result can be obtained by an application of the Girsanov Theorem. We propose a detailed proof because it introduces important ideas that are used in the sequel.

Let us define the N random vectors Y^j ($j = -n, \cdots, n$) of $\mathcal{S} = \mathbb{R}^{T+1}$ by

$$\left\{\begin{array}{lcl} Y_0^j & = & \Psi_0(U^J) \\ Y_t^j & = & B_{t-1}^j + \bar{\theta} \quad j = -n, \cdots, n \quad t = 1, \cdots, T. \end{array}\right.$$

It can be seen that

$$Y^j \simeq \mathcal{N}_{T+1}(\bar{\theta}\mathrm{OZ}_{T+1}, \sigma^2 \mathrm{Id}_{T+1}) \quad j = -n, \cdots, n, \tag{21}$$

where OZ_{T+1} is the vector[5] of $\mathbb{R}^{T+1}$ whose first coordinate is equal to 0 and the last T are equal to 1. Note that the variables Y^j are mutually independent.

For fixed (J, θ), we let $R_{J,\theta} : \mathbb{R}^{N(T+1)} \to \mathbb{R}^{N(T+1)}$ be the mapping $u \to y$, i.e.

$$R_{J,\theta}(u^{-n}, \cdots, u^n) = (y^{-n}, \cdots, y^n)$$

such that

$$\begin{cases} y_0^j &= \Psi_0(u_0^j) \\ y_t^j &= u_t^j - \gamma u_{t-1}^j - \sum_{i=-n}^n J_{ji} f(u_{t-1}^i) - \theta_j + \bar{\theta} \quad t = 1, \cdots, T, \end{cases}$$

for $j = -n, \cdots, n$. Since the determinant D of the Jacobian of $R_{J,\theta}$ is equal to $\prod_{j=-n}^n \Psi_0'(u_0^j)$, which is non zero by definition of Ψ_0, $R_{J,\theta}$ is a bijection of $\mathbb{R}^{N(T+1)}$ into itself.

Let $\varphi \in C_b(\mathcal{S}^N)$ and let us compute

$$S(J, \theta) = \mathbb{E}\left[\varphi(U^{-n}, \cdots, U^n) \mid (J, \theta)\right] = \frac{1}{D}\mathbb{E}\left[\varphi(R_{J,\theta}^{-1}(Y^{-n}, \cdots, Y^n)) \mid (J, \theta)\right].$$

Since (Y^j) $(j = -n, \cdots, n)$ are independent of (J, θ), using (21) we write

$$S(J, \theta) = \int_{\mathcal{S}^N} \varphi(R_{J,\theta}^{-1}(y^{-n}, \cdots, y^n)) \frac{1}{D}$$
$$(2\pi\sigma^2)^{-\frac{N(T+1)}{2}} \exp - \frac{\sum_{j=-n}^n \left(\sum_{t=1}^T (y_t^j - \bar{\theta})^2 + (y_0^j)^2\right)}{2\sigma^2} \left(\prod_{j=-n}^n \prod_{t=0}^T dy_t^j\right).$$

Through the inverse change of variables
$(y^{-n}, \cdots, y^n) \to (u^{-n}, \cdots, u^n) = R_{J,\theta}^{-1}(y^{-n}, \cdots, y^n)$ we write

$$S(J, \theta) = \int_{\mathcal{S}^N} \varphi(u^{-n}, \cdots, u^n) \times$$
$$(2\pi\sigma^2)^{-\frac{NT}{2}} \exp - \frac{1}{2\sigma^2} \Phi^{(J,\theta)}(u^{-n}, \cdots, u^n) \prod_{j=-n}^n \underline{\mu}_I(du_0^j) \prod_{t=1}^T du_t^j.$$

[5] A more natural notation would have been ZO_{T+1} but we prefer the former in reference to the celebrated wizard.

Here,

$$\Phi^{(J,\theta)}(u^{-n},\cdots,u^n)=\sum_{j=-n}^{n}\sum_{t=1}^{T}\left(u_t^j-\gamma u_{t-1}^j-\bar{\theta}-G_t^j\right)^2=$$

$$\sum_{j=-n}^{n}\sum_{t=1}^{T}\left[(u_t^j-\gamma u_{t-1}^j-\bar{\theta})^2+(G_t^j)^2-2(u_t^j-\gamma u_{t-1}^j-\bar{\theta})G_t^j\right],$$

where we have used the definition in (20). After noting proposition 1, we find that

$$S(J,\theta)=\int_{\mathcal{S}^N}\varphi(u^{-n},\cdots,u^n)\psi^{(J,\theta)}(u^{-n},\cdots,u^n)\,\underline{P}^{\otimes N}(du^{-n}\cdots du^n),\quad (22)$$

where

$$\psi^{(J,\theta)}(u^{-n},\cdots,u^n)=\exp\left(\frac{1}{\sigma^2}\left(\sum_{j=-n}^{n}\langle\Psi(u^j),G^j\rangle-\frac{1}{2}\|G^j\|^2\right)\right).$$

Since (22) is true for all $\varphi\in C_b(\mathbb{R}^{N(T+1)})$ we conclude that

$$\frac{d\underline{Q}^N(J,\theta)}{d\underline{P}^{\otimes N}}(u^{-n},\cdots,u^n)=\exp\frac{1}{\sigma^2}\left(\sum_{j=-n}^{n}\langle\Psi(u^j),G^j\rangle-\frac{1}{2}\|G^j\|^2\right),$$

where $\underline{Q}^N(J,\theta)$ is the regular conditional probability of $\underline{Q}$ given (J,Θ). By taking the expected value with respect to (J,θ) we obtain (19). □

Note that the Radon-Nikodym derivative does not depend upon $\{\Psi_0(u^j)\}$, $j=-n,\cdots,n$. We could have worked with T-dimensional processes G^j at the cost of making the last part of the paper heavier on notation.

We now study the Gaussian system $(G_s^i)_{i=-n,\dots,n,s=0,\dots,T}$ in more detail. Throughout the rest of this section, we consider the $u\in\mathcal{S}$ (in terms of which the system is defined) to be fixed, as is $x\in\mathcal{T}$, where $x_s^j=f(u_s^j)$. It will be seen that we may write the mean and covariance of the system as a function of the empirical measure $\hat{\mu}^N(x)$ (defined in (14)). This is of crucial importance because it will mean that the image laws of $P^{\otimes N}$ and Q^N under $\hat{\mu}^N$, i.e. R^N and Π^N, have Radon-Nikodym derivative given by the push-forward of $\frac{dQ^N}{dP^{\otimes N}}$. For $\mu\in\mathcal{M}_1^+(\mathcal{T}^{\mathbb{Z}})$, we define

$$c_t^\mu=\begin{cases}0 & t=0\\ \bar{J}\int_{\mathcal{T}^{\mathbb{Z}}}y_{t-1}^0\,d\mu(y), & t=1,\dots,T.\end{cases}\quad (23)$$

Similarly, for $\mu^N \in \mathcal{M}_1^+(\mathcal{T}^N)$ let K^{μ^N} be the $N(T+1) \times N(T+1)$ block circulant matrix with ith block given by (for $i = -n, \dots, n$)

$$K_{ts}^{\mu^N, i} = \begin{cases} \theta^2 \delta_i + \sum_{m=-n}^{n} \Lambda^N(i,m) \int_{\mathcal{T}^N} y_{t-1}^0 y_{s-1}^m \, \mu^N(dy) \\ \qquad\qquad \text{for } s,\, t = 1, \cdots, T \\ 0 \qquad \text{otherwise.} \end{cases} \tag{24}$$

Proposition 4. *Fix $x \in \mathcal{T}^N$ and let $u \in \mathcal{S}^N$ be such that $x_s^j = f(u_s^j)$. The covariance of the Gaussian system (G_s^i), where $i = -n, \dots, n$ and $s = 0, \dots, T$ writes $K^{(\hat{\mu}^N(x))^N}$, where $(\hat{\mu}^N(x))^N$ is the N-dimensional marginal of $\hat{\mu}^N(x)$. For each i, the mean of G^i is $c^{\hat{\mu}^N(x)}$.*

Proof. The mean of G_t^i is 0 if $t = 0$, or otherwise is equal to

$$\mathbb{E}\left[G_t^i\right] = \frac{\bar{J}}{N} \sum_{j=-n}^{n} f(u_{t-1}^j) = \frac{\bar{J}}{N} \sum_{j=-n}^{n} x_{t-1}^j = \bar{J} \int_{\mathcal{T}^{\mathbb{Z}}} y_{t-1}^0 \, d\hat{\mu}^N(x)(y),$$

for $t = 1, \cdots, T$. This is indeed independent of the index i.

Let us now examine the covariance function K of these N Gaussian processes. It is an $N(T+1) \times N(T+1)$ matrix which has a block structure, each block K^{ik}, i, $k = -n, \cdots, n$, being the $(T+1) \times (T+1)$ covariance matrix of the two processes G^i and G^k. We have

$$K_{ts}^{ik} = 0,$$

if s or t is equal to 0. We deal with the case where s and t differ from 0 in the remaining of the proof, i.e

$$K_{ts}^{ik} = cov(G_t^i G_s^k) = \sum_{j,l=-n}^{n} cov(J_{ij} J_{kl}) x_{t-1}^j x_{s-1}^l + \theta^2 \delta_{i-k}, \; s, t = 1, \cdots, T. \tag{25}$$

Because of our definition (5) of the covariance structure we have

$$K_{ts}^{ik} = \sum_{m=-n}^{n} \Lambda^N(i-k, m) \left(\frac{1}{N} \sum_{j=-n}^{n} x_{t-1}^j x_{s-1}^{j+m} \right) + \theta^2 \delta_{i-k}.$$

Since K^{ik} depends only on $(i-k)$, it can be seen that K is a block circulant matrix, and we may write

$$K^{ik} \stackrel{\text{def}}{\equiv} K^{(i-k) \mod N},$$

where we recall that $j \mod N$ lies between $\pm n$. It follows from the symmetry of Λ^N in (4) that the matrix K^i is symmetric, i.e. $K^i_{ts} = K^i_{st}$. It also follows from (4) that $K^{-i} = K^i$. It may be inferred from (25) that

$$K^i_{ts} = \theta^2 \delta_i + \sum_{m=-n}^{n} \Lambda^N(i,m) \int_{\mathcal{T}^N} y^0_{t-1} y^m_{s-1} \, (\hat{\mu}^N(x))^N(dy).$$

□

We note that $K^{(\hat{\mu}^N(x))^N}$ is positive as (by definition) it is the covariance matrix of a Gaussian system. For $\mu^N \in \mathcal{M}^+_{1,s}(\mathcal{T}^N)$, we denote the discrete Fourier transform of $(K^{\mu^N,j})$, $j = -n, \ldots, n$ by $(\tilde{K}^{\mu^N,l})$, i.e. for $-n \le l \le n$,

$$\begin{aligned} \tilde{K}^{\mu^N,l} &= \sum_{j=-n}^{n} K^{\mu^N,j} e^{-\frac{2\pi i j l}{N}}, \\ K^{\mu^N,j} &= N^{-1} \sum_{l=-n}^{n} \tilde{K}^{\mu^N,l} e^{\frac{2\pi i j l}{N}}. \end{aligned} \tag{26}$$

We employ the convention of denoting the discrete Fourier transform of a sequence by a tilde throughout this paper. We state a basic result from the theory of block-circulant matrices, noting that the matrix indexing is from $-n, \ldots, n$.

Lemma 5. *Let B be a symmetric block-circulant matrix with the (j,k) $(T+1) \times (T+1)$ block given by $(B^{(j-k) \mod N})$, $j,k = -n, \cdots, n$. Let $\tilde{B}^k = \sum_{l=-n}^{n} B^l \exp\left(-\frac{2\pi i k l}{N}\right)$ for $|k| \le n$, and $W^{(N)}$ be the $N \times N$ Hermitian matrix with elements $W^{(N)}_{jk} = \frac{1}{\sqrt{N}} \exp(\frac{2\pi i j k}{N})$, $j,k = -n, \cdots, n$. Then B may be 'block'-diagonalised in the follow manner (where $\otimes$ is the Kronecker Product and * the complex conjugate),*

$$B = (W^{(N)} \otimes \mathrm{Id}_{T+1}) diag\left(\tilde{B}^{-n}, \ldots, \tilde{B}^{n}\right) (W^{(N)} \otimes \mathrm{Id}_{\mathrm{T+1}})^*.$$

We observe also that λ is an eigenvalue of B if and only if λ is an eigenvalue of $\tilde{B}^k$ for some k. Finally the sequence (B^j) $(j = -n, \ldots, n)$ is both real and even if and only if the sequence $(\tilde{B}^k)$ $(k = -n, \ldots, n)$ is both real and even.

We now provide another form of equation (19) by applying to it the following lemma from Gaussian calculus [34, 36] which we recall for completeness:

Lemma 6. *Let Z be a Gaussian vector of $\mathbb{R}^p$ with mean c and covariance matrix K. If $a \in \mathbb{R}^p$ and $b \in \mathbb{R}$ is such that for all eigenvalues α of K the relation $\alpha b > -1$ holds, we have*

$$\mathbb{E}\left[\exp\left({}^t aZ - \frac{b}{2}\|Z\|^2\right)\right] =$$
$$\frac{1}{\sqrt{\det\left(\mathrm{Id}_p + bK\right)}} \times \exp\left({}^t ac - \frac{b}{2}\|c\|^2 + \frac{1}{2}\,{}^t(a-bc)K\left(\mathrm{Id}_p + bK\right)^{-1}(a-bc)\right)$$

This leads us to the following proposition.

Proposition 7. *The Radon-Nikodym derivative of $\underline{Q}^N$ with respect to $\underline{P}^{\otimes N}$ is also given by the following expression,*

$$\frac{d\underline{Q}^N}{d\underline{P}^{\otimes N}}(u^{-n}, \cdots, u^n) = \frac{1}{\sqrt{\det\left(\mathrm{Id}_{N(T+1)} + \frac{1}{\sigma^2}K^{(\hat{\mu}^N(x))^N}\right)}} \times$$
$$\exp\left(\frac{1}{\sigma^2}\left(\sum_{i=-n}^{n}\left(\left\langle c^{\hat{\mu}^N(x)}, \Psi(u^i)\right\rangle - \frac{1}{2}\|c^{\hat{\mu}^N(x)}\|^2\right)\right.\right.$$
$$\left.\left. + \frac{1}{2}\sum_{i,j=-n}^{n}\left\langle \Psi(u^i) - c^{\hat{\mu}^N(x)}, A^{(\hat{\mu}^N(x))^N, ij}\left(\Psi(u^j) - c^{\hat{\mu}^N(x)}\right)\right\rangle\right)\right). \quad (27)$$

Here $A^{(\hat{\mu}^N)^N, ij}$, $i, j = -n, \cdots, n$ are defined to be the $(T+1) \times (T+1)$ blocks of the $N(T+1) \times N(T+1)$ matrix $K^{(\hat{\mu}^N)^N}(\sigma^2 \mathrm{Id}_{N(T+1)} + K^{(\hat{\mu}^N)^N})^{-1}$.

Proof. The eigenvalues of $K^{(\hat{\mu}^N(x))^N}$ are positive because it is a covariance matrix. We thus obtain our result by the application of lemma 6 to equation (19) with $p = N(T+1)$, $Z = (G^{-n}, G^{-n+1}, \cdots, G^n)$,
$a = \frac{1}{\sigma^2}(\Psi(u^{-n}), \Psi(u^{-n+1}), \cdots, \Psi(u^n))$, $i = -n, \cdots, n$, and $b = \frac{1}{\sigma^2}$. $\square$

We define the subset $\hat{\mathcal{M}}_{1,s}^{N,+}(\mathcal{T}^{\mathbb{Z}}) = \bigcup_{x \in \mathcal{T}^N} \hat{\mu}^N(x) \subset \mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$. The definition of $\hat{\mathcal{M}}_{1,s}^{N,+}(\mathcal{S}^{\mathbb{Z}})$ is analogous.

The righthand side of (27) is the product of two terms which we analyse in some detail in order to prepare the ground for the definition of a rate function.

We first note that the first term can obviously be rewritten as described in the following lemma.

Lemma 8. *The following relation holds:*

$$\frac{1}{\sqrt{\det\left(\mathrm{Id}_{N(T+1)} + \frac{1}{\sigma^2} K^{(\hat{\mu}^N(x))^N}\right)}} = \exp(N\Gamma_1((\hat{\mu}^N(x))^N)),$$

where for $\mu \in \hat{\mathcal{M}}_{1,s}^{N,+}(\mathcal{T}^{\mathbb{Z}})$, we define

$$\Gamma_1(\mu^N) = -\frac{1}{2N} \log\left(\det\left(\mathrm{Id}_{N(T+1)} + \frac{1}{\sigma^2} K^{\mu^N}\right)\right). \tag{28}$$

The above expression has sense because the eigenvalues of $\mathrm{Id}_{N(T+1)} + \sigma^{-2} K^{(\hat{\mu}^N(x))^N}$ are bounded below by 1. We next express the second term as a function of the empirical measure $\hat{\mu}^N$. This is done by elucidating the block structure of the matrix $A^{(\hat{\mu}^N(x))^N}$ as revealed in the following lemma.

Lemma 9. *The matrix $A^{(\hat{\mu}^N(x))^N} = K^{(\hat{\mu}^N(x))^N}(\sigma^2 \mathrm{Id}_{N(T+1)} + K^{(\hat{\mu}^N(x))^N})^{-1}$ is symmetric and block circulant. It is built from an even sequence of $(T+1) \times (T+1)$ symmetric matrixes $A^{(\hat{\mu}^N(x))^N, i}$, $i = -n, \cdots, n$. Furthermore we have*

$$\exp\left(\frac{1}{\sigma^2}\left(\sum_{i=-n}^{n}\left(\langle c^{\hat{\mu}^N(x)}, \Psi(u^i)\rangle - \frac{1}{2}\|c^{\hat{\mu}^N(x)}\|^2\right) + \frac{1}{2}\sum_{i,j=-n}^{n} \langle \Psi(u^i) - c^{\hat{\mu}^N(x)}, A^{(\hat{\mu}^N(x))^N, ij}(\Psi(u^j) - c^{\hat{\mu}^N(x)})\rangle\right)\right) = \exp(N\Gamma_2(\hat{\mu}^N(x))),$$

where for $\mu \in \hat{\mathcal{M}}_{1,s}^{N,+}(\mathcal{T}^{\mathbb{Z}})$, we define

$$\Gamma_2(\mu^N) = \frac{1}{2\sigma^2} \int_{\mathcal{S}^N} \left(\sum_{i=-n}^{n} \langle \Psi(v^0) - c^\mu, A^{\mu^N, i}(\Psi(v^i) - c^\mu)\rangle + 2\langle c^\mu, \Psi(v^0)\rangle - \|c^\mu\|^2\right) \underline{\mu}^N(u)(dv). \tag{29}$$

Proof. It can be seen that the matrix $A^{(\hat{\mu}^N(x))^N}$ is block-circulant through the diagonalisation of $K^{(\hat{\mu}^N(x))^N}$ (given in lemma 5) in the definition of $A^{(\hat{\mu}^N(x))^N}$. We index the blocks as $A^{(\hat{\mu}^N(x))^N, i}$, $i = -n, \cdots, n$, where

$$A^{(\hat{\mu}^N(x))^N, ij} = A^{(\hat{\mu}^N(x))^N, (i-j) \mod N}.$$

The diagonalisation in lemma 5 allows us to write

$$\tilde{A}^{(\hat{\mu}^N(x))^N,j} = \tilde{K}^{(\hat{\mu}^N(x))^N,j}\left(\sigma^2 \mathrm{Id}_{T+1} + \tilde{K}^{(\hat{\mu}^N(x))^N,j}\right)^{-1}.$$

This means that $\tilde{A}^{(\hat{\mu}^N(x))^N,j} = \tilde{A}^{(\hat{\mu}^N(x))^N,-j}$ and the blocks are symmetric (since these properties apply to the blocks of $\tilde{K}^{(\mu^N(x))^N}$). In turn, this means that $A^{(\hat{\mu}^N(x))^N,-j} = A^{(\hat{\mu}^N(x))^N,j}$ and ${}^tA^{(\hat{\mu}^N(x))^N,j} = A^{(\hat{\mu}^N(x))^N,j}$. The result now follows from a substitution of the definitions. □

It is useful to put together proposition 7, lemma 8 and lemma 9 in the following proposition.

Proposition 10. *The Radon-Nikodym derivative of $\underline{Q}^N$ with respect to $\underline{P}^{\otimes N}$ writes*

$$\frac{d\underline{Q}^N}{d\underline{P}^{\otimes N}}(u^{-n},\cdots,u^n) = \exp(N\Gamma((\hat{\mu}^N(f(u^{-n}),\cdots,f(u^n)))^N)),$$

where for $\mu \in \hat{\mathcal{M}}_{1,s}^{N,+}(\mathcal{T}^{\mathbb{Z}})$, $\Gamma(\mu^N) = \Gamma_1(\mu^N) + \Gamma_2(\mu^N)$ and the expressions for Γ_1 and Γ_2 are given by lemmas 8 and 9.

Before we close this section we define a subset of $\mathcal{M}_{1,s}^{+}$ which appears naturally.

Definition 4. We define the subset $\mathcal{E}_2$ of $\mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$ by

$$\mathcal{E}_2 = \{\mu \in \mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}}) \,|\, \mathbb{E}^{\underline{\mu}}[\|\Psi(u^0)\|^2] < \infty\}.$$

For this set of measures, we may define the stationary process $(v^k)_{k\in\mathbb{Z}}$ in $\mathcal{S}^{\mathbb{Z}}$, where $v^k = \Psi(u^k)$. This has a finite mean $\mathbb{E}^{\underline{\underline{\mu}}}[v^0]$, noted $\bar{v}^\mu$, where we recall from definition 2 that $\underline{\underline{\mu}} = \underline{\mu} \circ \Psi^{-1}$. It admits the following spectral density measure, noted $\tilde{v}^\mu$, such that

$$\mathbb{E}^{\underline{\underline{\mu}}}[v^0\, {}^tv^k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\omega}\, \tilde{v}^\mu(d\omega). \tag{30}$$

We similarly define

$$\mathcal{E}_2^{(N)} = \{\mu \in \mathcal{M}_{1,s}^{+}(\mathcal{T}^N) \,|\, \mathbb{E}^{\underline{\mu}}[\|\Psi(u^0)\|^2] < \infty\},$$

and note that if $\mu \in \mathcal{E}_2$ then $\mu^N \in \mathcal{E}_2^{(N)}$.

4 The image of the averaged law through the empirical measure

In the previous section we saw that the Radon-Nikodym derivative of Q^N with respect to $P^{\otimes N}$ may be expressed as a function of the empirical measure, i.e. $\exp(N\Gamma((\hat{\mu}^N(x))^N))$. In this section we obtain an expression for the Radon-Nikodym derivative of the image laws Π^N and R^N under the empirical measure. We do this by extending Γ beyond the range of $\hat{\mu}^N$ to an arbitrary measure in $\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$. We will see that $\frac{d\Pi^N}{dR^N}(\mu)$ is $\exp(N\Gamma(\mu^N))$, and that the extended function $\Gamma(\mu^N)$ is lower semi-continuous.

4.1 Gaussian processes

We determine the Radon-Nikodym derivative at μ^N by writing Γ as a function of a Gaussian process G^{μ^N} which is, in turn, determined by μ^N. We begin with finite N, before proceeding to the infinite-dimensional projective limit.

4.1.1 Finite number of neurons

Given μ^N in $\mathcal{M}_{1,s}^+(\mathcal{T}^N)$ we define the stationary $N(T+1)$-dimensional Gaussian process G^{μ^N}. We will use G^{μ^N} to define $\Gamma(\mu^N)$.

In analogy to (23), the mean of $G_t^{\mu^N,i}$ is equal to 0 if $t=0$, or otherwise

$$c_t^{\mu^N} = \bar{J}\int_{\mathcal{T}^N} y_{t-1}^i \mu^N(dy),\ t=1,\cdots,T,\ i=-n,\cdots,n.$$

We note that the above integral is independent of i due to the stationarity of μ^N, which is why we have omitted the superscript i from $c_t^{\mu^N}$.

For each $\mu^N \in \mathcal{M}_{1,s}^+(\mathcal{T}^N)$ let $M^{\mu^N,k}$, $(k=-n,\cdots,n)$, be the $(T+1)\times(T+1)$ matrix defined by (for $s,t\in[1,T]$),

$$M_{st}^{\mu^N,k} = \int_{\mathcal{T}^N} y_{s-1}^0 y_{t-1}^k \mu^N(dy). \tag{31}$$

If $s=0$ or $t=0$, then $M_{st}^{\mu^N,k}=0$.

The covariance matrix of G^{μ^N} is defined by the block circulant $N(T+1)\times N(T+1)$ matrix K^{μ^N}, which has blocks given (in analogy with equation (24)) by

$$K_{st}^{\mu^N,i} = \theta^2\delta_i(1-\delta_s)(1-\delta_t) + \sum_{m=-n}^{n}\Lambda(i,m)M_{st}^{\mu^N,m}, \tag{32}$$

for $i=-n,\cdots,n$. The discrete Fourier transform $(\tilde{K}^{\mu^N,j})$of the sequence $(K^{\mu^N,i})$ is given by (26).

We now state some properties of the matrices we have just defined. We will prove that K^{μ^N} is positive, which means that it is a well-defined covariance matrix.

Lemma 11. *For all $k=-n,\cdots,n$,*

$$^tM^{\mu^N,k}=M^{\mu^N,-k}. \tag{33}$$

The blocks $(K^{\mu^N,k})$ are symmetric and satisfy $K^{\mu^N,-k}=K^{\mu^N,k}$. Furthermore $\tilde{M}^{\mu^N,k}$ (for all $k=-n,\cdots,n$), $\tilde{K}^{\mu^N,k}$ (for all $k=-n,\cdots,n$) and K^{μ^N} are all positive.

Proof. The identity (33) follows directly from the stationarity of μ^N. The evenness of the series $(K^{\mu^N,i})$, $i=-n,\ldots,n$, follows from (4) and (32). It follows from (33) that

$$K_{st}^{\mu^N,i}=\theta^2\delta_i(1-\delta_s)(1-\delta_t)+\sum_{m=-n}^{n}\Lambda(i,m)M_{ts}^{\mu^N,-m}.$$

We find, in turn, by (4), that

$$K_{st}^{\mu^N,i}=\theta^2\delta_i(1-\delta_s)(1-\delta_t)+\sum_{m=-n}^{n}\Lambda(i,m)M_{ts}^{\mu^N,m}=K_{ts}^{\mu^N,i},$$

which means that $K^{\mu^N,i}$ is symmetric.

Let M^{μ^N} be the block-circulant matrix with blocks given by $M^{\mu^N,k}$, $k=-n,\ldots,n$. Let $\tau:\mathcal{T}^N\to\mathcal{T}^N$ be the map such that for all $|k|\le n$, $\tau(x)_0^k=0$ and for $s\in[1,T]$, $\tau(x)_s^k=x_{s-1}^k$. It may be observed from (31) that M^{μ^N} is the correlation matrix of $\mu\circ\tau^{-1}$, which means that it is positive. It then follows from lemma 5 that the matrixes

$$\tilde{M}^{\mu^N,l}=\sum_{k=-n}^{n}M^{\mu^N,k}e^{-\frac{2\pi ikl}{N}},$$

are positive. We also observe from this lemma that

$$M^{\mu^N,k}=\frac{1}{N}\sum_{l=-n}^{n}\tilde{M}^{\mu^N,l}e^{\frac{2\pi ikl}{N}}. \tag{34}$$

To prove that the matrix K^{μ^N} is positive, it suffices (by lemma 5) to prove that $\tilde{K}^{\mu^N,l}$ is positive for all $l = (-n, \ldots, n)$. Using (34), we write

$$\tilde{K}^{\mu^N,l} = \theta^2 \mathrm{OZ}_{T+1}\,{}^t\mathrm{OZ}_{T+1} + \frac{1}{N}\sum_{p=-n}^{n}\left(\sum_{m=-n}^{n}\sum_{k=-n}^{n}\Lambda(k,m)e^{-\frac{2\pi i(kl-mp)}{N}}\right)\tilde{M}^{\mu^N,p}.$$

Using the symmetry $\Lambda(k,-m) = \Lambda(k,m)$, this can be rewritten in terms of the spectral density $\tilde{\Lambda}^N$ of Λ, i.e.

$$\tilde{K}^{\mu^N,l} = \theta^2 \mathrm{OZ}_{T+1}\,{}^t\mathrm{OZ}_{T+1} + \frac{1}{N}\sum_{p=-n}^{n}\tilde{\Lambda}^N(l,p)\tilde{M}^{\mu^N,p}.$$

Hence, for $W \in \mathcal{S}$, we have

$$ {}^tW\tilde{K}^{\mu^N,l}W = \theta^2\langle \mathrm{OZ}_{T+1}, W\rangle^2 + \frac{1}{N}\sum_{p=-n}^{n}\tilde{\Lambda}^N(l,p)\left({}^tW\tilde{M}^{\mu^N,p}W\right). \tag{35}$$

This is positive because the spectral density $\tilde{\Lambda}^N$ is positive and ${}^tW\tilde{M}^{\mu^N,p}W$ is positive. $\square$

4.1.2 Infinite number of neurons

Given μ in $\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ we define a stationary Gaussian process G^μ with values in $\mathcal{S}^{\mathbb{Z}}$. It will be seen that G^μ is the limit of G^{μ^N} (where μ^N is the N-dimensional marginal of μ) as $N \to \infty$, in the sense that the means and covariances converge.

The mean is the same as the finite-dimensional case. That is, for all i the mean of $G^{\mu,i}_t$ is given by c^μ_t, where

$$c^\mu_t = \bar{J}\int_{\mathcal{T}^{\mathbb{Z}}} y^i_{t-1}d\mu(y),\ t = 1, \cdots, T\,, i \in \mathbb{Z},\ \text{ and } c^\mu_0 = 0, \tag{36}$$

the above integral being independent of i due to the stationarity of μ.

We now define the covariance of G^μ. The definition of $M^{\mu,k}$ is analogous to the previous definition, i.e.

Definition 5. Let $M^{\mu,k}$, $k \in \mathbb{Z}$ be the $(T+1)\times(T+1)$ matrix defined by (for $s, t \in [1, T]$),

$$M^{\mu,k}_{st} = \int_{\mathcal{T}^{\mathbb{Z}}} y^0_{s-1}y^k_{t-1}d\mu(y). \tag{37}$$

If $s = 0$ or $t = 0$, then $M^{\mu,k}_{st} = 0$.

These matrixes satisfy ${}^t M^{\mu,k} = M^{\mu,-k}$ because of the stationarity of μ. Furthermore, they feature a spectral representation, i.e. there exists a $(T+1) \times (T+1)$ matrix-valued measure $\tilde{M}^\mu = (\tilde{M}^\mu)_{s,t=0,\cdots,T}$ with the following properties. Each $\tilde{M}^\mu_{st}$ is a complex measure on $[-\pi, \pi[$ of finite total variation and such that

$$M^{\mu,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\omega} \tilde{M}^\mu(d\omega). \tag{38}$$

Furthermore, for all vectors $W \in \mathbb{R}^{T+1}$, ${}^t W \tilde{M}(d\omega) W$ is a positive measure on $[-\pi, \pi[$.

The covariance between elements $G^{\mu,i}$ and $G^{\mu,i+k}$ is defined to be

$$K^{\mu,k} = \theta^2 \delta_k \mathrm{OZ}_{T+1} {}^t\mathrm{OZ}_{T+1} + \sum_{l=-\infty}^{\infty} \Lambda(k,l) M^{\mu,l}. \tag{39}$$

We note that the above summation converges for all $k \in \mathbb{Z}$ since the series $(\Lambda(k,l))_{k,l \in \mathbb{Z}}$ is absolutely convergent and the elements of $M^{\mu,l}$ are bounded by ± 1 for all $l \in \mathbb{Z}$. We next prove that the sequence $(K^{\mu,k})_{k \in \mathbb{Z}}$ admits a spectral representation (which in turn implies that K^μ is a well-defined covariance operator).

Proposition 12. *The sequence $(K^{\mu,k})_{k \in \mathbb{Z}}$ has spectral density $\tilde{K}^\mu$ given by*

$$\tilde{K}^\mu(\omega) = \theta^2 \mathrm{OZ}_{T+1} {}^t\mathrm{OZ}_{T+1} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Lambda}(\omega, \gamma) \tilde{M}(d\gamma).$$

That is, $\tilde{K}^\mu$ is positive and satisfies

$$K^{\mu,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\omega} \tilde{K}^\mu(\omega) d\omega.$$

Proof. First we prove that the matrix function

$$\tilde{K}^\mu(\omega) = \sum_{k=-\infty}^{\infty} K^{\mu,k} e^{-ik\omega}$$

is well-defined on $[-\pi, \pi[$ and is equal to the expression in the statement of the proposition. Afterwards, we will prove that $\tilde{K}^\mu$ is positive.

From (39) we obtain that, for all $s, t \in [0, T]$,

$$|K_{st}^{\mu,k}| \leq T\theta^2 \delta_k + \sum_{l=-\infty}^{\infty} |\Lambda(k,l)|. \tag{40}$$

This shows that, because by (7) the series $(\Lambda(k,l))_{k,l\in\mathbb{Z}}$ is absolutely convergent, $\tilde{K}^{\mu}(\omega)$ is well-defined on $[-\pi, \pi[$.

Using (38) we write

$$\tilde{K}^{\mu}(\omega) = \theta^2 \mathrm{OZ}_{T+1}\, {}^t\mathrm{OZ}_{T+1} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \Lambda(k,m) e^{-i(k\omega - m\gamma)} \right) \tilde{M}^{\mu}(d\gamma).$$

Using the symmetry $\Lambda(k,-m) = \Lambda(k,m)$ this can be rewritten in terms of the spectral density $\tilde{\Lambda}$ of Λ

$$\tilde{K}^{\mu}(\omega) = \theta^2 \mathrm{OZ}_{T+1}\, {}^t\mathrm{OZ}_{T+1} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\Lambda}(\omega,\gamma) \tilde{M}^{\mu}(d\gamma).$$

We note that $\tilde{K}^{\mu}(\omega)$ is positive, because for all vectors W of $\mathbb{R}^{T+1}$,

$${}^tW \tilde{K}^{\mu}(\omega) W = \theta^2 \langle \mathrm{OZ}_{T+1}, W \rangle^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\Lambda}(\omega,\gamma) \left({}^tW \tilde{M}^{\mu}(d\gamma) W \right),$$

the spectral density $\tilde{\Lambda}$ is positive and the measure ${}^tW \tilde{M}^{\mu}(d\gamma) W$ is positive.

□

The finite-dimensional system 'converges' to the infinite-dimensional system in the following sense. In what follows, we use the Frobenius norm on the (T+1)-dimensional matrices. We write $\tilde{K}^{\mu^N}(\omega) = \sum_{k=-n}^{n} K^{\mu^N,k} \exp(-ik\omega)$. Note that for $|j| \leq n$, $\tilde{K}^{\mu^N}(2\pi j/N) = \tilde{K}^{\mu^N,j}$. The lemma below follows directly from the absolute convergence of $\sum_{j,k} |\Lambda(j,k)|$.

Lemma 13. *Fix $\mu \in \mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$. For all ε, there exists an N such that for all $M > N$ and all j such that $2|j|+1 \leq M$, $\|K^{\mu^M,j} - K^{\mu,j}\| < \varepsilon$ and for all $\omega \in [-\pi, \pi[$, $\|\tilde{K}^{\mu^M}(\omega) - \tilde{K}^{\mu}(\omega)\| \leq \varepsilon$.*

Lemma 14. *The eigenvalues of $\tilde{K}^{\mu^N,l}$ and $\tilde{K}^{\mu}(\omega)$ are upperbounded by*

$$\rho_K \stackrel{\text{def}}{\equiv} (T+1)\left(\theta^2 + \Lambda^{sum}\right),$$

where Λ^{sum} is defined in (7).

Proof. Let $W \in \mathcal{S}$. In the finite-dimensional case, we find from (35), (6), and (7) that

$$
\begin{aligned}
{}^{t}W\tilde{K}^{\mu^N,l}W &\leq \theta^2(T+1)\|W\|^2 + \Lambda^{sum}\frac{1}{N}\sum_{p=-n}^{n}\left({}^{t}W\tilde{M}^{\mu^N,p}W\right). \\
&= \theta^2(T+1)\|W\|^2 + \Lambda^{sum}\,{}^{t}WM^{\mu^N,0}W.
\end{aligned}
$$

The eigenvalues of $M^{\mu^N,0}$ are all positive (since it is a covariance matrix), which means that each eigenvalue is upperbounded by the trace, which in turn is upperbounded by $T+1$. Through taking the limit $N \to \infty$ we also obtain the upperbound for $\tilde{K}^{\mu}(\omega)$. □

We let $A^{\mu^N} = K^{\mu^N}(\sigma^2 \mathrm{Id}_{N(T+1)} + K^{\mu^N})^{-1}$. This is well-defined because K^{μ^N} is diagonalizable (being symmetric and real) and has positive eigenvalues. Furthermore, it follows from lemma 5 that this is even block circulant, with symmetric blocks $A^{\mu^N,k}$ $(k = -n, \cdots, n)$ and that

$$
\tilde{A}^{\mu^N,l} = \sum_{k=-n}^{n} A^{\mu^N,k} e^{-\frac{2\pi i k l}{N}} = \tilde{K}^{\mu^N,l}(\sigma^2 \mathrm{Id}_{T+1} + \tilde{K}^{\mu^N,l})^{-1}. \tag{41}
$$

In the limit $N \to \infty$ we may define

$$
\tilde{A}^{\mu}(\omega) = \tilde{K}^{\mu}(\omega)(\sigma^2 \mathrm{Id}_{T+1} + \tilde{K}^{\mu}(\omega))^{-1}
$$

as the product of two functions defined on $[-\pi, \pi[$ whose Fourier series are absolutely convergent. The Fourier series of $(\sigma^2 \mathrm{Id}_{T+1} + \tilde{K}^{\mu}(\omega))^{-1}$ is absolutely convergent as a consequence of Wiener's theorem because the eigenvalues of $\sigma^2 \mathrm{Id}_{T+1} + \tilde{K}^{\mu}(\omega)$ are strictly positive. Hence the Fourier series of $\tilde{A}^{\mu}(\omega)$, i.e. $(A^{\mu,k})_{k\in\mathbb{Z}}$, is absolutely convergent. We thus find that, for $l \in \mathbb{Z}$,

$$
A^{\mu,l} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{A}^{\mu}(\omega)e^{il\omega}d\omega = \lim_{N\to\infty} A^{\mu^N,l}, \tag{42}
$$

and

$$
\tilde{A}^{\mu}(\omega) = \sum_{l=-\infty}^{\infty} A^{\mu,l}e^{-il\omega}.
$$

Let $\tilde{A}^{\mu^N}(\omega) = \sum_{k=-n}^{n} A^{\mu^N,k}\exp(-ik\omega)$ and note that for $|j| \leq n$,
$\tilde{A}^{\mu^N}(2\pi j/N) = \tilde{A}^{\mu^N,j}$.

Lemma 15. *The map* $B \to B(\sigma^2 \mathrm{Id}_{T+1} + B)^{-1}$ *is Lipschitz over the set* $\Delta = \{\tilde{K}^{\mu^N}(\omega), \tilde{K}^{\mu}(\omega) : \mu \in \mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}}), N > 0, \omega \in [-\pi, \pi[\}$. *That is, there exists a positive constant* A_{lip} *such that for all* $B_1, B_2 \in \Delta$,

$$\|B_1(\sigma^2 \mathrm{Id}_{T+1} + B_1)^{-1} - B_2(\sigma^2 \mathrm{Id}_{T+1} + B_2)^{-1}\| \leq A_{lip} \|B_1 - B_2\|.$$

Proof. The eigenvalues λ of the matrixes in Δ satisfy $0 \leq \lambda \leq \rho_K$. Thus, both B and $(\sigma^2 \mathrm{Id}_{T+1} + B)^{-1}$ are bounded in the operator norm (which is equal to the largest eigenvalue) for all $B \in \Delta$. They are thus bounded over every matrix norm (as the matrix norms are all equivalent). The first term is clearly Lipschitz, and the second term is also Lipschitz because

$$\begin{aligned}(\sigma^2 \mathrm{Id}_{T+1} + B_1)^{-1} &- (\sigma^2 \mathrm{Id}_{T+1} + B_2)^{-1} \\ &= (\sigma^2 \mathrm{Id}_{T+1} + B_1)^{-1} (B_2 - B_1) (\sigma^2 \mathrm{Id}_{T+1} + B_2)^{-1}.\end{aligned}$$

□

The following lemma is a consequence of lemmas 13 and 15.

Lemma 16. *Fix* $\mu \in \mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$. *For all* ε, *there exists an* N *such that for all* $M > N$ *and all* $\omega \in [-\pi, \pi[$, $\|\tilde{A}^{\mu^M}(\omega) - \tilde{A}^{\mu}(\omega)\| \leq \varepsilon$.

The above-defined matrices have the following 'uniform convergence' properties.

Proposition 17. *Fix* $\nu \in \mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$. *For all* $\varepsilon > 0$, *there exists an open neighbourhood* $V_\varepsilon(\nu)$ *such that for all* $\mu \in V_\varepsilon(\nu)$, *all* $s, t \in [0, T]$ *and all* $\omega \in [-\pi, \pi[$,

$$\left| \tilde{K}^{\nu}_{st}(\omega) - \tilde{K}^{\mu}_{st}(\omega) \right| \leq \varepsilon, \tag{43}$$

$$\left| \tilde{A}^{\nu}_{st}(\omega) - \tilde{A}^{\mu}_{st}(\omega) \right| \leq \varepsilon, \tag{44}$$

$$|c^{\nu}_s - c^{\mu}_s| \leq \varepsilon, \tag{45}$$

and for all $N > 0$, *and for all* k *such that* $|k| \leq n$,

$$\left| \tilde{K}^{\nu^N, k}_{st} - \tilde{K}^{\mu^N, k}_{st} \right| \leq \varepsilon, \tag{46}$$

and

$$\left| \tilde{A}^{\nu^N, k}_{st} - \tilde{A}^{\mu^N, k}_{st} \right| \leq \varepsilon. \tag{47}$$

Proof. The bounds are evident if $s = 0$ or $t = 0$ as the elements are all zero, hence we may assume that s and t are nonzero throughout this proof. Let μ be in $\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ and $\omega \in [-\pi, \pi[$. We have

$$\tilde{K}^{\mu}_{st}(\omega) - \tilde{K}^{\nu}_{st}(\omega) = \sum_{k=-\infty}^{\infty} (K^{\mu,k}_{st} - K^{\nu,k}_{st}) e^{-ik\omega}.$$

Using (39) we have

$$\tilde{K}^{\mu}_{st}(\omega) - \tilde{K}^{\nu}_{st}(\omega) = \sum_{k,l=-\infty}^{\infty} \Lambda(k,l)(M^{\mu,l}_{st} - M^{\nu,l}_{st}) e^{-ik\omega},$$

hence

$$\left|\tilde{K}^{\mu}_{st}(\omega) - \tilde{K}^{\nu}_{st}(\omega)\right| \leq \sum_{k,l=-\infty}^{\infty} |\Lambda(k,l)| \int_{\mathcal{T}^{2L}} \left|x^0_{s-1}x^l_{t-1} - y^0_{s-1}y^l_{t-1}\right| \mathcal{L}^{2L}(dx, dy),$$

where $L = 2|l| + 1$ and $\mathcal{L}^{2L}$ has marginals μ^L and ν^L. Since $|x^0_{s-1}x^l_{t-1} - y^0_{s-1}y^l_{t-1}| \leq 2d_L(\pi_L x, \pi_L y)$, we find (through (16)) that

$$\left|\tilde{K}^{\mu}_{st}(\omega) - \tilde{K}^{\nu}_{st}(\omega)\right| \leq 2d(\mu, \nu).$$

Thus for (43) to be satisfied, it suffices for us to stipulate that $V_{\varepsilon}(\nu)$ is a ball of radius less than $\frac{1}{2}\epsilon$ (with respect to the distance metric in (16)). Similar reasoning dictates that (46) is satisfied too.

However in light of lemma 15, it is evident that we may take the radius of $V_{\varepsilon}(\nu)$ to be sufficiently small that (43), (46) and (47) are satisfied. In fact (44) is also satisfied, as it may be obtained by taking the limit as $N \to \infty$ of (47). Since c^{μ} is determined by the one-dimensional marginal of μ, it follows from the definition of the metric in (16) that we may take the radius of $V_{\varepsilon}(\nu)$ to be sufficiently small that (45) is satisfied too. □

A direct consequence of the above proposition is that $c^{\mu}, \tilde{K}^{\mu^N}, \tilde{K}^{\mu}, \tilde{A}^{\mu^N}$ and $\tilde{A}^{\mu}$ are continuous with respect to μ.

4.2 Definition of the functional Γ

We have previously (in (8) and (29)) defined a functional $\Gamma := \Gamma_1 + \Gamma_2$ on the image of $\hat{\mu}^N$. We now extend these definitions to functionals Γ_1, Γ_2 :

$\mathcal{M}_{1,s}^{+}(\mathcal{T}^N) \to \mathbb{R}$. It will be seen that these functionals asymptote to a limit as $N \to \infty$, so that we may consider Γ_1 and Γ_2 to be defined on $\mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$ as well.

Let $\mu \in \mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$, and let $(\mu^N)_{N\geq 1}$ be the N-dimensional marginals of μ (for $N = 2n+1$ odd).

4.2.1 Γ_1

We define

$$\Gamma_1(\mu^N) = -\frac{1}{2N}\log\left(\det\left(\mathrm{Id}_{N(T+1)} + \frac{1}{\sigma^2}K^{\mu^N}\right)\right). \tag{48}$$

Because of lemma 11 the spectrum of K^{μ^N} is positive, that of $\mathrm{Id}_{N(T+1)} + \frac{1}{\sigma^2}K^{\mu^N}$ is strictly positive and the above expression has a sense. Moreover, $\Gamma_1(\mu^N) \leq 0$.

We now define $\Gamma_1(\mu) = \lim_{N\to\infty}\Gamma_1(\mu^N)$. The following lemma indicates that this is well-defined.

Lemma 18. *When N goes to infinity the limit of (48) is given by*

$$\Gamma_1(\mu) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\left(\det\left(\mathrm{Id}_{T+1} + \frac{1}{\sigma^2}\tilde{K}^{\mu}(\omega)\right)\right)d\omega \tag{49}$$

for all $\mu \in \mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$.

Proof. Through lemma 5, we have that

$$\Gamma_1(\mu^N) = -\frac{1}{2N}\sum_{l=-n}^{n}\log\left(\det\left(\mathrm{Id}_{T+1} + \sigma^{-2}\tilde{K}^{\mu^N}\left(\frac{2\pi l}{N}\right)\right)\right), \tag{50}$$

where we recall that $\tilde{K}^{\mu^N}\left(\frac{2\pi l}{N}\right) = \tilde{K}^{\mu^N,l}$. Since, by lemma 13, $\tilde{K}^{\mu^N}(\omega)$ converges uniformly to $\tilde{K}^{\mu}(\omega)$, it is evident that the above expression converges to the desired result. ∎

Proposition 19. Γ_1 *is bounded below and continuous on both $\mathcal{M}_{1,s}^{+}(\mathcal{T}^N)$ and $\mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$.*

Proof. Applying lemma 6 in the case of $Z = (G^{\mu^N,-n} - c^{\mu^N}, \cdots, G^{\mu^N,n} - c^{\mu^N})$, $a = 0$, $b = \sigma^{-2}$, we write

$$\Gamma_1(\mu^N) = \frac{1}{N} \log \mathbb{E}\left[\exp\left(-\frac{1}{2\sigma^2} \sum_{k=-n}^{n} \|G^{\mu^N,k} - c^{\mu^N}\|^2\right)\right].$$

Using Jensen's inequality we have

$$\Gamma_1(\mu^N) \geq -\frac{1}{2N\sigma^2} \mathbb{E}\left[\sum_{k=-n}^{n} \|G^{\mu^N,k} - c^{\mu^N}\|^2\right] = -\frac{1}{2\sigma^2} \mathbb{E}\left[\|G^{\mu^N,0} - c^{\mu^N}\|^2\right].$$

By definition of $K^{\mu^N,0}$, the righthand side is equal to $-\frac{1}{2\sigma^2}\mathrm{Trace}(K^{\mu^N,0})$. From (32), we find that

$$\mathrm{Trace}(K^{\mu^N,0}) = T\theta^2 + \sum_{m=-n}^{n} \Lambda(0,m)\mathrm{Trace}(M^{\mu^N,m}).$$

It follows from the definition (31) that

$$0 \leq |\mathrm{Trace}(M^{\mu^N,m})| \leq T.$$

We obtain

$$\mathrm{Trace}(K^{\mu^N,0}) \leq T\left(\theta^2 + \sum_{m=-n}^{n} |\Lambda(0,m)|\right) \leq T\left(\theta^2 + \Lambda^{sum}\right)$$

Hence

$$\Gamma_1(\mu^N) \geq -\beta_1,$$

where

$$\beta_1 = \frac{T}{2\sigma^2}\left(\theta^2 + \Lambda^{sum}\right). \tag{51}$$

It follows from lemma 18 that $-\beta_1$ is a lower bound for $\Gamma_1(\mu)$ as well.

The continuity (over both $\mathcal{M}^+_{1,s}(\mathcal{T}^N)$ and $\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$) follows from the expressions (48) and (49), continuity of the applications $\mu^N \to \tilde{K}^{\mu^N}$ and $\mu \to \tilde{K}^{\mu}$ (proposition 17) and the continuity of the determinant. $\square$

4.2.2 Γ_2

We define, analogously to (29),

$$\Gamma_2(\mu^N) = \frac{1}{2\sigma^2} \int_{\mathcal{S}^N} \left(\sum_{k=-n}^{n} \sum_{t,\, s=0}^{T} A_{ts}^{\mu^N,\, k} (v_t^0 - c_t^{\mu^N})(v_s^k - c_s^{\mu^N}) + \right.$$
$$\left. 2 \left\langle c^{\mu^N}, v^0 \right\rangle - \|c^{\mu^N}\|^2 \right) \underline{\underline{\mu}}^N(dv), \quad (52)$$

with $\|c^{\mu^N}\|^2 = \sum_{t=0}^{T} \left(c_t^{\mu^N} \right)^2$ and $\underline{\underline{\mu}}$ defined in definition 2. This quantity is finite in the subset $\mathcal{E}_2^N$ of $\mathcal{M}_{1,s}^+(\mathcal{T}^N)$ defined in definition 4. If $\mu^N \notin \mathcal{E}_2^N$, then we set $\Gamma_2(\mu^N) = \infty$.

We define
$$\Gamma_2(\mu) = \lim_{N \to \infty} \Gamma_2(\mu^N),$$
where μ^N is the N-dimensional marginal of μ. If $\mu \notin \mathcal{E}_2$ then $\mu^N \notin \mathcal{E}_2^N$ and $\Gamma_2(\mu) = \infty$. We assume throughout the rest of this section that $\mu \in \mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$ is in $\mathcal{E}_2$. This means that the spectral measure $\tilde{v}^\mu$ (as given in (30)) exists. The following proposition indicates that $\Gamma_2(\mu)$ is well-defined.

Proposition 20. *If the measure μ is in $\mathcal{E}_2$, i.e. if $\mathbb{E}^{\underline{\underline{\mu}}}[\|v^0\|^2] < \infty$, then $\Gamma_2(\mu)$ is finite and writes*

$$\Gamma_2(\mu) = \frac{1}{2\sigma^2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{A}^\mu(\omega) : \tilde{v}^\mu(d\omega) \right.$$
$$\left. + {}^t c^\mu (\tilde{A}^\mu(0) - \mathrm{Id}_{T+1}) c^\mu + 2\mathbb{E}^{\underline{\underline{\mu}}} \left[{}^t v^0 (\mathrm{Id}_{T+1} - \tilde{A}^\mu(0)) c^\mu \right] \right).$$

The ":" symbol indicates the double contraction on the indexes. One also has

$$\Gamma_2(\mu) = \frac{1}{2\sigma^2} \left(\lim_{n \to \infty} \sum_{k=-n}^{n} \int_{\mathcal{S}^{\mathbb{Z}}} {}^t (v^0 - c^\mu) A^{\mu,\, k} (v^k - c^\mu) \, d\underline{\underline{\mu}}(v) \right.$$
$$\left. + 2\mathbb{E}^{\underline{\underline{\mu}}}[\langle c^\mu, v^0 \rangle] - \|c^\mu\|^2 \right).$$

Proof. We note firstly that $c^{\mu^N} = c^{\mu}$. Using (30), the stationarity of μ and the fact that $\sum_{k=-n}^{n} A^{\mu^N,k} = \tilde{A}^{\mu^N}(0)$, we have

$$\Gamma_2(\mu^N) = \frac{1}{4\pi\sigma^2}\int_{-\pi}^{\pi}\sum_{k=-n}^{n}\exp(ik\omega)A^{\mu^N,k} : \tilde{v}^{\mu}(d\omega)$$
$$+\frac{1}{\sigma^2}\int_{\mathcal{S}^{\mathbb{Z}}}\langle c^{\mu}, v^0\rangle - {}^t c^{\mu}\tilde{A}^{\mu^N}(0)v^0 d\underline{\underline{\mu}}(v) + \frac{1}{2\sigma^2}{}^t c^{\mu}\left(\mathrm{Id}_{T+1} - \tilde{A}^{\mu^N}(0)\right)c^{\mu}. \quad (53)$$

From the spectral representation of A^{μ^N} we find that

$$\Gamma_2(\mu^N) = \frac{1}{4\pi\sigma^2}\int_{-\pi}^{\pi}\tilde{A}^{\mu^N}(\omega) : \tilde{v}^{\mu}(d\omega)$$
$$+\frac{1}{\sigma^2}E^{\underline{\underline{\mu}}}\left[{}^t v^0(\mathrm{Id}_{T+1} - \tilde{A}^{\mu^N}(0))c^{\mu}\right] + \frac{1}{2\sigma^2}{}^t c^{\mu}\left(\mathrm{Id}_{T+1} - \tilde{A}^{\mu^N}(0)\right)c^{\mu}. \quad (54)$$

Since (according to proposition 16) $\tilde{A}^{\mu^N}(\omega)$ converges uniformly to $\tilde{A}^{\mu}(\omega)$ as $N \to \infty$, it follows by dominated convergence that $\Gamma_2(\mu^N)$ converges to the expression in the proposition.

The second expression for $\Gamma_2(\mu)$ follows analogously, although this time we make use of the fact that the partial sums of the Fourier Series of $\tilde{A}^{\mu}$ converge uniformly to $\tilde{A}^{\mu}$ (because the Fourier Series is absolutely convergent). $\square$

We next obtain more information about the eigenvalues of the matrices $\tilde{A}^{\mu^N,k} = \tilde{A}^{\mu^N}(\frac{2k\pi}{N})$ (where $k = -n, \ldots, n$) and $\tilde{A}^{\mu}(\omega)$.

Lemma 21. *There exists $\alpha < 1$, such that for all N, μ and ω, the eigenvalues of $\tilde{A}^{\mu^N,k}$, $\tilde{A}^{\mu}(\omega)$ and A^{μ^N} are less than or equal to α.*

Proof. By lemma 14, the eigenvalues of $\tilde{K}^{\mu}(\omega)$ are positive and upperbounded by ρ_K. Since $\tilde{K}^{\mu}(\omega)$ and $\left(\sigma^2\mathrm{Id}_{T+1} + \tilde{K}^{\mu}(\omega)\right)^{-1}$ are coaxial (because $\tilde{K}^{\mu}$ is real and symmetric and therefore diagonalisable), we may take

$$\alpha = \frac{\rho_K}{\sigma^2 + \rho_K}.$$

This upperbound also holds for $\tilde{A}^{\mu^N,k}$, and for the eigenvalues of A^{μ^N} because of lemma 5. $\square$

We wish to prove that $\Gamma_2(\mu^N)$ is lower semicontinuous. A consequence of this will be that $\Gamma_2(\mu^N)$ is measureable with respect to $\mathcal{B}(\mathcal{M}_{1,s}(\mathcal{T}^N))$. In order to do this, we must first prove that its integrand possesses a lower bound. We do this by diagonalising A^{μ^N} into its spectral representation.

We use the fact that the measure μ^N is stationary to rewrite (52) in a more symmetric fashion,

$$\Gamma_2(\mu^N) = \frac{1}{2\sigma^2}\int_{\mathcal{S}^N}\left(\frac{1}{N}\sum_{k=-n}^{n}\sum_{j=-n}^{n} {}^t(v^j - c^{\mu^N})A^{\mu^N,k}(v^{k+j} - c^{\mu^N}) + \frac{2}{N}\sum_{j=-n}^{n}\langle v^j - c^{\mu^N}, c^{\mu^N}\rangle + \|c^{\mu^N}\|^2\right)\underline{\underline{\mu}}^N(dv), \quad (55)$$

where we recall that the neuron-indexing is taken modulo N. Define the N $(T+1)$-dimensional vectors

$$w^k = v^k - c^{\mu^N}, \quad k = -n, \cdots, n.$$

We use (41) to replace $A^{\mu^N,k}$ by its Fourier representation, so that the quadratic component of the above expression becomes

$$\frac{1}{N}\sum_{k=-n}^{n}\sum_{j=-n}^{n} {}^t w^j A^{\mu^N,k} w^{k+j} = \frac{1}{N^2}\sum_{j,k,l=-n}^{n} {}^t w^j \tilde{A}^{\mu^N,l} w^{k+j} e^{\frac{2\pi i k l}{N}}.$$

Using the shift property of the Discrete Fourier Transform we write

$$\sum_{k=-n}^{n} w^{k+j} e^{\frac{2\pi i k l}{N}} = \tilde{w}^{l*} e^{-\frac{2\pi i j l}{N}},$$

where the $*$ denotes the complex conjugate. Finally we obtain

$$\frac{1}{N}\sum_{j,k=-n}^{n} {}^t w^j A^{\mu^N,l} w^{k+j} = \frac{1}{N^2}\sum_{l=-n}^{n} {}^t\tilde{w}^l \tilde{A}^{\mu^N,l}\tilde{w}^{l*}.$$

We notice that each of the terms in the above summation is positive. Indeed for all l $\tilde{A}^{\mu^N,l}$ is real and symmetric positive, hence

$${}^t\tilde{w}^l \tilde{A}^{\mu^N,l}\tilde{w}^{l*} = {}^t\mathrm{Re}(\tilde{w}^l)\tilde{A}^{\mu^N,l}\mathrm{Re}(\tilde{w}^l) + {}^t\mathrm{Im}(\tilde{w}^l)\tilde{A}^{\mu^N,l}\mathrm{Im}(\tilde{w}^l).$$

The linear term $\frac{2}{N}\sum_{j=-n}^{n}\langle v^j - c^{\mu^N}, c^{\mu^N}\rangle$ writes $\frac{2}{N}\ \langle \tilde{w}^0, c^{\mu^N}\rangle$. We conclude that the integrand in the definition of $\Gamma_2(\mu^N)$ is equal to $1/(2\sigma^2)$ times

$$\frac{1}{N^2}\sum_{l=-n}^{n} {}^t\tilde{w}^l \tilde{A}^{\mu^N,l}\tilde{w}^{l*} + \frac{2}{N}\langle \tilde{w}^0, c^{\mu^N}\rangle + \|c^{\mu^N}\|^2. \tag{56}$$

In order to show that this expression is bounded below, it is sufficient to show that

$$\frac{1}{N^2}\,{}^t\tilde{w}^0\tilde{A}^{\mu^N,0}\tilde{w}^0 + \frac{2}{N}\ \langle \tilde{w}^0, c^{\mu^N}\rangle, \tag{57}$$

is bounded below, where we have made use of the fact that $\tilde{w}^0$ is real. Let $\tilde{K}^{\mu^N,0} = O^{\mu^N}D^{\mu^N}\,{}^tO^{\mu^N}$, where D^{μ^N} is diagonal and O^{μ^N} is orthonormal. We define $X ={}^t O^{\mu^N}\tilde{w}^0$, so that (57) is equal to

$$\frac{1}{N^2}\,{}^tXD^{\mu^N}(\sigma^2\mathrm{Id}_{T+1} + D^{\mu^N})^{-1}X + \frac{2}{N}\sum_{t=0}^{T}\langle {}^tO_t^{\mu^N}, c^{\mu^N}\rangle X_t, \tag{58}$$

where $O_t^{\mu^N}$ is the t-th column vector of O^{μ^N}. In order that (58) is bounded below, we require that the coefficient of X converges to zero when D^{μ^N} does. The following lemma is sufficient.

Lemma 22. *For each $0 \le t \le T$,*

$$\langle c^{\mu^N}, O_t^{\mu^N}\rangle^2 \le \frac{\bar{J}^2}{\tilde{\Lambda}^{min}} D_{tt}^{\mu^N}.$$

Proof. If $\bar{J} = 0$ the conclusion is evident, thus we assume throughout this proof that $\bar{J} \neq 0$. It follows from the definition that

$$\tilde{K}^{\mu^N,0} = \sum_{m=-n}^{n} K^{\mu^N,m}.$$

Expressing $K^{\mu^N,m}$ in terms of the matrixes $M^{\mu^N,k}$ we write

$$\tilde{K}^{\mu^N,0} = \theta^2 \mathrm{OZ}_{T+1}\,{}^t\mathrm{OZ}_{T+1} + \sum_{k,m=-n}^{n}\Lambda^N(m,k)M^{\mu^N,k}.$$

Since $D_{tt}^{\mu^N} = {}^t\bar{O}_t^{\mu^N}\tilde{K}^{\mu^N,0}O_t^{\mu^N}$, we find that

$$D_{tt}^{\mu^N} = \theta^2\langle \mathrm{OZ}_{T+1}, O_t^{\mu^N}\rangle^2 + \sum_{k,m=-n}^{n}\Lambda^N(k,m)\,{}^tO_t^{\mu^N}M^{\mu^N,k}O_t^{\mu^N}.$$

We introduce the matrixes $(L^{\mu^N,k})_{k=-n,\cdots,n}$, where for $1 \le s, t \le T$,

$$L^{\mu^N,k}_{st} = M^{\mu^N,k}_{st} - \bar{c}^\mu_s \bar{c}^\mu_t = \int_{\mathcal{T}^N} (y^0_{s-1} - \bar{c}^\mu_{s-1})(y^k_{t-1} - \bar{c}^\mu_{t-1})\, \mu^N(dy)$$

where $\bar{c}^\mu = \frac{1}{\bar{J}} c^{\mu^N}$. We define $L^{\mu^N,k}_{st} = 0$ if $s = 0$ or $t = 0$.

These matrices have the same properties as the matrixes $M^{\mu^N,k}$, in particular their spectral representation $(\tilde{L}^{\mu^N,l})_{l=-n,\cdots,n}$ is positive. Using this spectral representation we write

$$D^{\mu^N}_{tt} = \theta^2 \langle \mathrm{OZ}_{T+1}, O^{\mu^N}_t \rangle^2 + \tilde{\Lambda}^N(0,0) \langle \bar{c}^\mu, O^{\mu^N}_t \rangle^2 + \frac{1}{N} \sum_{l=-n}^{n} \tilde{\Lambda}^N(0,-l)\, {}^t O^{\mu^N}_t \tilde{L}^{\mu^N,l} O^{\mu^N}_t,$$

and since $\tilde{\Lambda}^N(0,-l)$ is positive for all $l = -n, \cdots, n$ and ${}^t O^{\mu^N}_t \tilde{L}^{\mu^N,l} O^{\mu^N}_t$ is positive for all $t = 1, \cdots, T$, we have

$$D^{\mu^N}_{tt} \ge \frac{\tilde{\Lambda}^N(0,0)}{\bar{J}^2} \langle c^{\mu^N}, O^{\mu^N}_t \rangle^2,$$

and the conclusion follows from assumption (8).

□

We may use the previous lemma to obtain a lower-bound for the quadratic form (58). We recall the easily-proved identity from the calculus of quadratics that, for all $x \in \mathbb{R}$,

$$ax^2 + 2bx \ge -\frac{b^2}{a}.$$

We therefore find, through lemma 22, that (58) is greater than or equal to

$$-\frac{\bar{J}^2}{\tilde{\Lambda}^{\min}} \left((T+1)\sigma^2 + \sum_{t=0}^{T} D^{\mu^N}_{tt} \right) = -\frac{\bar{J}^2}{\tilde{\Lambda}^{\min}} \left((T+1)\sigma^2 + \mathrm{Trace}(\tilde{K}^{\mu^N,0}) \right). \tag{59}$$

Since $\tilde{K}^{\mu^N,0} = \sum_{k=-n}^{n} K^{\mu,k}$ and

$$K^{\mu,k} = \theta^2 \delta_k \mathrm{OZ}_{T+1}\, {}^t\mathrm{OZ}_{T+1} + \sum_{m=-n}^{n} \Lambda(k,m) M^{\mu,m},$$

it follows that

$$\mathrm{Trace}(\tilde{K}^{\mu^N,0}) \le (T+1) \left(\theta^2 + \Lambda^{sum} \right).$$

Putting all this together we find that the integrand of $\Gamma_2(\mu^N)$ (we denote this ϕ^N further below) is greater than $-\beta_2$, where

$$\beta_2 = \frac{(T+1)\bar{J}^2}{2\sigma^2\tilde{\Lambda}^{\min}}\left(\sigma^2+\theta^2+\Lambda^{sum}\right). \tag{60}$$

Note that we have 'recollected' the factor of $1/2\sigma^2$. This is a 'universal' constant which depends only on the model parameters and not on the particular measure μ.

We then have the following proposition.

Proposition 23. $\Gamma_2(\mu^N)$ *is lower-semicontinous.*

Proof. We take the integrand of (55) and 'shift' it up, so that it is positive. That is, we define

$$\phi^N(\mu^N,v) = \beta_2 + \frac{1}{2\sigma^2}\Bigg(\frac{2}{N}\sum_{j=-n}^{n}\langle c^{\mu^N},(v^j-c^{\mu^N})\rangle + \|c^{\mu^N}\|^2 + \\ \frac{1}{N}\sum_{k=-n}^{n}\sum_{j=-n}^{n} {}^t(v^j-c^{\mu^N})A^{\mu^N,k}(v^{k+j}-c^{\mu^N})\Bigg), \tag{61}$$

which, as we have just proved, is greater than or equal to zero. We define $\phi^{N,M}(\mu^N,v) = 1_{B_M}\phi^N(\mu^N,v)$, where $v\in B_M$ if $N^{-1}\sum_{j=-n}^{n}\|v^j\|^2 \le M$. We also define

$$\Gamma_2^M(\mu^N) = \int_{\mathcal{S}^N}\phi^{N,M}(\mu,v)\underline{\underline{\mu}}^N(dv) - \beta_2.$$

Suppose that $\mu_k^N \to \mu^N$ with respect to the weak topology. Observe that

$$\left|\Gamma_2^M(\mu^N)-\Gamma_2^M(\mu_k^N)\right| \le \left|\int_{\mathcal{S}^N}\phi^{N,M}(\mu^N,v)\underline{\underline{\mu}}^N(dv) - \int_{\mathcal{S}^N}\phi^{N,M}(\mu^N,v)\underline{\underline{\mu}}_k^N(dv)\right| \\ + \left|\int_{\mathcal{S}^N}\phi^{N,M}(\mu^N,v)\underline{\underline{\mu}}_k^N(dv) - \int_{\mathcal{S}^N}\phi^{N,M}(\mu_k^N,v)\underline{\underline{\mu}}_k^N(dv)\right|.$$

We may infer from the above expression that $\Gamma_2^M(\mu^N)$ is continuous (with respect to μ^N) for the following reasons. The first term on the right hand side converges to zero because $\phi^{N,M}$ is continuous and bounded (with respect to

v). The second term converges to zero because $\phi^{N,M}(\mu^N, v)$ is a continuous function of μ^N, see proposition 17.

Since $\Gamma_2^M(\mu^N)$ grows to $\Gamma_2(\mu^N)$ as $M \to \infty$, we may conclude that $\Gamma_2(\mu^N)$ is lower semicontinuous with respect to μ^N. $\square$

We define $\Gamma(\mu^N) = \Gamma_1(\mu^N) + \Gamma_2(\mu^N)$. We may conclude from propositions 19 and 23 that Γ is measureable. It thus follows from definition 3 and proposition 10 that

Corollary 24. *The Radon-Nikodym derivative of* Π^N *with respect to* R^N *is given by*

$$\frac{d\Pi^N}{dR^N}(\mu) = \exp(N\Gamma(\mu^N)),$$

where μ^N *denotes the* N*-dimensional marginal of* μ*.*

We have the following alternative expression for $\Gamma(\mu^N)$.

Proposition 25. $\Gamma(\mu^N)$ *is given by the following expression*

$$\Gamma(\mu^N) = \frac{1}{N}\int_{\mathcal{S}^N} \log \mathbb{E}\left[\prod_{i=-n}^{n} \exp\left(\frac{1}{\sigma^2}\langle v^i, G^{\mu^N,i}\rangle - \frac{1}{2}\|G^{\mu^N,i}\|^2\right)\right] \underline{\underline{\mu}}^N(dv). \quad (62)$$

Proof. This is a matter of applying lemma 6 in the case of $Z = (G^{\mu^N,-n}, \cdots, G^{\mu^N,n})$, $a = \frac{1}{\sigma^2}(v^{-n}, \cdots, v^n)$, and $b = \frac{1}{\sigma^2}$, using the expression (48) for $\Gamma_1(\mu^N)$, and the fact that the measure $\mu^N \in \mathcal{M}^+_{1,s}(\mathcal{T}^N)$. $\square$

5 The large deviation principle

In this section we prove the principle result of this paper (Theorem 2), that the image laws Π^N satisfy an LDP with good rate function H (to be defined below). We do this by firstly establishing an LDP for the image law with uncoupled weights (R^N), and then use the Radon-Nikodym derivative of corollary 24 to establish the full LDP. Therefore our first task is to write the LDP governing R^N.

Let $\mu \in \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$. The Küllback-Leibler divergence, noted $I^{(2)}(\mu^N, P^{\otimes N})$, of μ^N with respect to $P^{\otimes N} = (P^{\mathbb{Z}})^N$ is defined as

$$I^{(2)}(\mu^N, P^{\otimes N}) = \int_{\mathcal{T}^N} \log\left(\frac{d\mu^N}{dP^{\otimes N}}\right) \frac{d\mu^N}{dP^{\otimes N}}\, dP^{\otimes N},$$

if μ^N is absolutely continuous with respect to $P^{\otimes N}$, and $I^{(2)}(\mu^N, P^{\otimes N}) = \infty$ otherwise. The process-level entropy of μ with respect to $P^{\mathbb{Z}}$ is defined to be

$$I^{(3)}(\mu, P^{\mathbb{Z}}) = \lim_{N\to\infty} \frac{1}{N} I^{(2)}(\mu^N, P^{\otimes N}). \tag{63}$$

R^N is governed by the following large deviation principle [21, 2]. If F is a closed set, then

$$\overline{\lim_{N\to\infty}} N^{-1} \log R^N(F) \leq - \inf_{\mu\in F} I^{(3)}(\mu, P^{\mathbb{Z}}),$$

and for all open sets O

$$\underline{\lim_{N\to\infty}} N^{-1} \log R^N(O) \geq - \inf_{\mu\in O} I^{(3)}(\mu, P^{\mathbb{Z}}).$$

We note the following two properties of $I^{(3)}$.

Lemma 26. *$I^{(3)}$ is a good rate function (i.e. its level sets are compact). In addition, the set of measures $\{R^N\}$ is exponentially tight. This means that, for all $0 \leq a < \infty$, there exists a compact set $K_a \subset M^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ such that for all N*

$$\overline{\lim_{N\to\infty}} N^{-1} \log R^N\left(K_a^c\right) < -a.$$

Proof. The fact that $I^{(3)}$ is a good rate function is proved in Ellis [23]. In turn, a sequence of probability measures (such as $\{R^N\}$) over a Polish Space satisfying a large deviations upper bound with a good rate function is exponentially tight [19]. □

Before we move to a statement of the LDP governing Π^N, we prove the following relationship between the set $\mathcal{E}_2$ (see definition 4) and the set of stationary measures which have a finite Küllback-Leibler information or process level entropy with respect to $P^{\mathbb{Z}}$.

Lemma 27. *We have*

$$\{\mu \in \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}}),\ I^{(3)}(\mu, P^{\mathbb{Z}}) < \infty\} \subset \mathcal{E}_2.$$

Proof. Let $\mu \in \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$. We use the classical result that

$$I^{(2)}(\mu^N, P^{\otimes N}) = \sup_{\varphi\in \mathrm{C}_b(\mathcal{T}^N)} \left(\int_{\mathcal{T}^N} \varphi \, d\mu^N - \log \int_{\mathcal{T}^N} \exp(\varphi) \, dP^{\otimes N} \right).$$

We let $\rho(y) = \sum_{k=-n}^{n} \|y^k\|^2$ and $\varphi = a\rho(\Psi(f^{-1}(x^{-n})), \ldots, \Psi(f^{-1}(x^n)))$, where Ψ is given in (12) and $a > 0$. The function $\rho_M(x) = \rho(x)\mathbf{1}_{\varphi(x)\leq M}$ is in $\mathrm{C}_b(\mathcal{T}^N)$, hence for all $a > 0$

$$a \int_{\mathcal{T}^N} \rho_M \, d\mu^N \leq \log \int_{\mathcal{T}^N} \exp(a\rho_M) \, dP^{\otimes N} + I^{(2)}(\mu^N, P^{\otimes N}).$$

According to proposition 1, $\underline{P} \circ \Psi^{-1} \simeq \mathcal{N}(0_{T+1}, \sigma^2 \mathrm{Id}_{T+1})$. Hence, as soon as $1 - 2a\sigma^2 > 0$, we obtain using an easy Gaussian computation that

$$\log \int_{\mathcal{T}^N} \exp(a\rho) \, dP^{\otimes N} = -\frac{N(T+1)}{2} \log(1 - 2a\sigma^2).$$

By dominated convergence, letting $M \to \infty$, we write

$$a \int_{\mathcal{T}^N} \rho \, d\mu^N \leq \log \int_{\mathcal{T}^N} \exp(a\rho) \, dP^{\otimes N} + I^{(2)}(\mu^N, P^{\otimes N}).$$

Hence, since $\int_{\mathcal{T}^N} \rho \, d\mu^N = N\mathbb{E}^{\underline{\mu}}[\|v^0\|^2]$, we have

$$a\mathbb{E}^{\underline{\mu}}[\|v^0\|^2] \leq -\frac{(T+1)}{2} \log(1 - 2a\sigma^2) + \frac{I^{(2)}(\mu^N, P^{\otimes N})}{N}.$$

By taking the limit $N \to \infty$ we obtain the result. $\square$

We are now in a position to define what will be the rate function of the LDP governing Π^N.

Definition 6. Let H be the function $\mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}}) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$H(\mu) = \begin{cases} +\infty & \text{if} \quad I^{(3)}(\mu, P^{\mathbb{Z}}) = \infty \\ I^{(3)}(\mu, P^{\mathbb{Z}}) - \Gamma(\mu) & \text{otherwise.} \end{cases}$$

Note that because of proposition 20 and lemma 27, whenever $I^{(3)}(\mu, P^{\mathbb{Z}})$ is finite, so is $\Gamma(\mu)$. It also needs to be noted that, for all N and $x \in \mathcal{T}^N$, $\hat{\mu}^N(x) \in \mathcal{E}_2$.

Our proof of the principal result, Theorem 2, of this paper will occur in several steps. We prove in sections 5.1 and 5.3 that Π^N satisfies a weak LDP, i.e. that it satisfies (17) when F is compact and (18) for all open O. We also prove in section 5.2 that $\{\Pi^N\}$ is exponentially tight, and we prove in section 5.4 that H is a good rate function. It directly follows from these results that Π^N satisfies a strong LDP with good rate function H [19]. Finally, in section 6 we prove that H has a unique minimum which $\hat{\mu}^N$ converges to weakly as $N \to \infty$.

5.1 Lower bound on the open sets

We prove the second half of proposition 2.

Lemma 28. *For all open sets O,*

$$\varliminf_{N\to\infty} N^{-1}\log \Pi^N(O) \geq -\inf_{\mu\in O} H(\mu).$$

Proof. From the expression for the Radon-Nikodym derivative in corollary 24 we have

$$\Pi^N(O) = \int_O \exp\left(N\Gamma(\mu^N)\right) dR^N(\mu).$$

If $\mu \in O$ is such that $I^{(3)}(\mu, P^{\mathbb{Z}}) = \infty$, then $H(\mu) = \infty$ and evidently

$$\varliminf_{N\to\infty} N^{-1}\log \Pi^N(O) \geq -H(\mu). \tag{64}$$

We now prove (64) for all $\mu \in O$ such that $I^{(3)}(\mu, P^{\mathbb{Z}}) < \infty$. Let $\epsilon > 0$ and $Z_\varepsilon^N(\mu) \subset O$ be an open neighbourhood containing μ such that $\inf_{\gamma\in Z_\varepsilon^N(\mu)} \Gamma(\gamma^N) \geq \Gamma(\mu^N) - \epsilon$. Such $\{Z_\varepsilon^N(\mu)\}$ exist for all N because of the lower semi-continuity of $\Gamma(\mu^N)$ (see proposition 23) and the fact that the projection $\mu \to \mu^N$ is clearly continuous. Then

$$\begin{aligned}
\varliminf_{N\to\infty} N^{-1}\log \Pi^N(O) &= \varliminf_{N\to\infty} N^{-1}\log \int_O \exp(N\Gamma(\gamma^N))dR^N(\gamma)\\
&\geq \varliminf_{N\to\infty} N^{-1}\log\left(R^N(Z_\varepsilon^N)\times \inf_{\gamma\in Z_\varepsilon^N(\mu)}\exp(N\Gamma(\gamma^N))\right)\\
&\geq -I^{(3)}(\mu, P^{\mathbb{Z}}) + \varliminf_{N\to\infty} \inf_{\gamma\in Z_\varepsilon^N(\mu)} \Gamma(\gamma^N)\\
&\geq -I^{(3)}(\mu, P^{\mathbb{Z}}) + \varliminf_{N\to\infty} \Gamma(\mu^N) - \epsilon\\
&= -I^{(3)}(\mu,)P^{\mathbb{Z}} + \Gamma(\mu) - \epsilon.
\end{aligned}$$

The last equality follows from lemma 18 and proposition 20. Since ϵ is arbitrary, we may take the limit as $\epsilon \to 0$ to obtain (64). Since (64) is true for all $\mu \in O$ the lemma is proved. $\square$

5.2 Exponential Tightness of Π^N

We recall that if $\mu \in \mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$ but $\mu \notin \mathcal{E}_2$, then $I^{(3)}(\mu, P^{\mathbb{Z}}) = \Gamma(\mu) = \infty$. Otherwise, $I^{(3)}$ and Γ satisfy the following affine inequality.

Proposition 29. *There exist constants $a > 1$ and $c > 0$ such that for all $\mu \in \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}}) \cap \mathcal{E}_2$,*

$$\Gamma(\mu) \le \frac{\left(I^{(3)}(\mu, P^{\mathbb{Z}}) + c\right)}{a}.$$

We have (from (63)) that

$$I^{(3)}(\mu, P^{\mathbb{Z}}) = \lim_{N\to\infty} N^{-1} I^{(2)}\left(\mu^N, P^{\otimes N}\right).$$

We recall that $I^{(2)}$ may be expressed using the Fenchel-Legendre transform as

$$I^{(2)}(\mu^N, P^{\otimes N}) = \sup_{\phi^N \in C_b(\mathcal{T}^N)} \left(\int_{\mathcal{T}^N} \phi^N(x)\mu^N(dx) - \log \int_{\mathcal{T}^N} \exp(\phi^N(x)) P^{\otimes N}(dx) \right), \quad (65)$$

where ϕ^N is a continuous, bounded function on $\mathcal{T}^N$. We choose a specific function ϕ^N to be N times the sum of $\Gamma_1(\mu^N)$ and the integrand of $\Gamma_2(\mu^N)$ (i.e. (55)) to which we add the constant $\beta = \beta_1 + \beta_2$ to make it positive. In detail, $\phi^N(x) = \underline{\underline{\phi}}^N(v)$, where $v^j = \Psi(f^{-1}(x^j))$ and

$$\underline{\underline{\phi}}^N(v) = N\left(\Gamma_1(\mu^N) + \beta\right) + \frac{1}{2\sigma^2}\left(\sum_{j,k=-n}^{n} {}^t(v^j - c^{\mu^N}) A^{\mu^N,k}(v^{j+k} - c^{\mu^N}) + 2\sum_{j=-n}^{n} \langle c^{\mu^N}, v^j\rangle - N\|c^{\mu^N}\|^2 \right), \quad (66)$$

and $\beta = \beta_1 + \beta_2$. β_1 and β_2 are defined in equations (51) and (60), respectively.

ϕ^N is continuous but not bounded in general. Hence we multiply it by $1_{\|v\|^2 \le M}(x)$ to obtain

$$\phi^N_M(x) = \phi^N(x) 1_{\|v\|^2 \le M}(x),$$

which is continuous and bounded.

It follows from (65) that, for all $a \ge 0$,

$$a \int_{\mathcal{T}^N} \phi^N_M(x)\mu^N(dx) \le \log \int_{\mathcal{T}^N} \exp\left(a\phi^N_M(x)\right) P^{\otimes N}(dx) + I^{(2)}(\mu^N, P^{\otimes N}). \quad (67)$$

We wish to use the dominated convergence theorem to prove that (67) holds in the limit as $M \to \infty$.

We may do this using the following lemma,

Lemma 30. *There exists a positive constant $c < \infty$ and $a > 1$ such that, for all N,*

$$\int_{\mathcal{T}^N} \exp(a\phi^N(x)) P^{\otimes N}(dx) \leq \exp(Nc + aN\beta).$$

Proof. We find from proposition 1 that

$$\int_{\mathcal{T}^N} \exp(a\phi^N(x)) P^{\otimes N}(dx) = \int_{\mathcal{S}^N} \exp(a\underline{\underline{\phi}}^N(v)) \underline{\underline{P}}^{\otimes N}(dv).$$

We use the spectral representation of (56) to rewrite the third term in (66) as

$$\frac{1}{N} \sum_{l=-n}^{n} {}^t\tilde{w}^l \tilde{A}^{\mu^N,l} \tilde{w}^{l*} + 2\langle c^\mu, \tilde{w}^0 \rangle + N\|c^\mu\|^2,$$

where

$$\tilde{w}^l = \sum_{k=-n}^{n} w^k e^{-\frac{2\pi i k l}{N}} \quad l = -n, \cdots, n,$$

and $w^k = v^k - c^{\mu^N}, \quad k = -n, \cdots, n$. Because the sequence $(v^k - c^{\mu^N})_{k=-n\cdots n}$ is real, the real part $(\mathrm{Re}(\tilde{w}^l))_{l=-n\cdots n}$ of the sequence $(\tilde{w^l})_{l=-n\cdots n}$ is even and the imaginary part $(\mathrm{Im}(\tilde{w}^l))_{l=-n\cdots n}$, odd. We perform the bjective affine change of variables in $\mathcal{S}^N$

$$h : (v^{-n}, \cdots, v^n) \to (y^{-n}, \cdots, y^n),$$

where, for $s \in [0, T]$,

$$y_s^k = \begin{cases} \sqrt{2}\mathrm{Re}(\tilde{w}_s^{-k}) & k = -1, \cdots, -n \\ \tilde{w}_s^0 & k = 0 \\ \sqrt{2}\mathrm{Im}(\tilde{w}_s^k) & k = 1, \cdots, n \end{cases}.$$

Moreover, the sequence $(A^{\mu^N,k})_{k=-n,\cdots,n}$ is symmetric even, which implies that

$$\sum_{l=-n}^{n} {}^t\tilde{w}^l \tilde{A}^{\mu^N,l} \tilde{w}^{l*} = \sum_{l=-n}^{n} {}^t y^l \tilde{A}^{\mu^N,l} y^l.$$

We thus find that

$$\underline{\underline{\phi}}^N(h^{-1}(y)) = N\Gamma_1(\mu) + N\beta + \frac{1}{2\sigma^2}\Bigg(\frac{1}{N}\,{}^t y^0 \tilde{A}^{\mu^N,0} y^0 + 2\langle c^\mu, y^0\rangle + N\|c^\mu\|^2 + \frac{1}{N}\sum_{|l|=1}^{n} {}^t y^l \tilde{A}^{\mu^N,l} y^l\Bigg).$$

Under h, it is easy to check, using the properties of the Discrete Fourier Transform, that

$$\sum_{j=-n}^{n} \|v^j\|^2 = \frac{1}{N}\left(\sum_{|j|=1}^{n} \|y^j\|^2 + \|y^0 + Nc^\mu\|^2\right).$$

It follows that

$$\underline{\underline{P}}^{\otimes N} \circ h^{-1}(dy) = \left(\sqrt{2\pi N\sigma^2}\right)^{-N(T+1)} \times \exp\left(-\frac{1}{2N\sigma^2}\left(\|y^0 + Nc^{\mu^N}\|^2 + \sum_{|j|=1}^{n} \|y^j\|^2\right)\right) \prod_{t=0}^{T}\prod_{j=-n}^{n} dy_t^j. \quad (68)$$

Hence we write

$$\int_{\mathcal{S}^N} \exp(a\underline{\underline{\phi}}^N(v))\underline{\underline{P}}^{\otimes N}(dv) = \int_{\mathcal{S}^N} \exp(a\underline{\underline{\phi}}^N(h^{-1}(y)))\underline{\underline{P}}^{\otimes N} \circ h^{-1}(dy) = \exp(aN(\Gamma_1(\mu^N) + \beta)) \times G_1 \times G_2,$$

where

$$G_1 = \left(\sqrt{2\pi N\sigma^2}\right)^{-(T+1)} \int_{\mathcal{S}} \exp\Bigg[\frac{1}{2N\sigma^2}\times \Big[a^t y^0 \tilde{A}^{\mu^N,0} y^0 + 2aN\langle c^{\mu^N}, y^0\rangle + aN^2\|c^{\mu^N}\|^2 - \|y^0 + Nc^{\mu^N}\|^2\Big]\Bigg] \prod_{t=0}^{T} dy_t^0$$

and

$$G_2 = \left(\sqrt{2\pi N\sigma^2}\right)^{-(N-1)(T+1)} \int_{\mathcal{S}^{(N-1)}} \exp\frac{1}{2N\sigma^2}\left[\sum_{|j|=1}^{n} a^t y^j \tilde{A}^{\mu^N,j} y^j - \|y^j\|^2\right] \prod_{|j|=1}^{n}\prod_{t=0}^{T} dy_t^j.$$

We assume that $a > 1$ is such that $(1 - a\alpha) > 0$, where α is an upperbound for the eigenvalues of A^{μ^N} given in Lemma 21. We find

$$G_2 \leq \left(\sqrt{2\pi N\sigma^2}\right)^{-(N-1)(T+1)}$$

$$\int_{\mathcal{S}^{(N-1)}} \exp \frac{1}{2N\sigma^2} \left[\sum_{|j|=1}^{n} (\alpha a - 1)\|y^j\|^2\right] \prod_{|j|=1}^{n} \prod_{t=0}^{T} dy_t^j$$

$$= \prod_{|j|=1}^{n} \mathbb{E}^{Y^j} \left[\exp\left(\frac{\alpha a}{2N\sigma^2}\|Y^j\|^2\right)\right].$$

where

$$Y^j \sim \mathcal{N}_{T+1}(0_{T+1}, N\sigma^2 \mathrm{Id}_{T+1}), \quad |j| = 1, \cdots, n.$$

The application of lemma 6 yields

$$\mathbb{E}^{Y_j} \left[\exp\left(\frac{\alpha a}{2N\sigma^2}\|Y^j\|^2\right)\right] = (1 - \alpha a)^{-(T+1)/2},$$

and

$$G_2 \leq (1 - a\alpha)^{-(N-1)(T+1)/2}.$$

Similarly

$$G_1 \leq \exp N \frac{(a-1)\|c^{\mu^N}\|^2}{2\sigma^2} \times \mathbb{E}^{Y^0} \left[\frac{\alpha a}{2N\sigma^2}\|Y^0\|^2 + \frac{a-1}{\sigma^2}\langle Y^0, c^{\mu^N}\rangle\right],$$

where

$$Y^0 \sim \mathcal{N}_{T+1}(0_{T+1}, N\sigma^2 \mathrm{Id}_{T+1}).$$

Another application of lemma 6 yields

$$G_1 \leq \exp N \frac{(a-1)\|c^{\mu^N}\|^2}{2\sigma^2} \times (1 - \alpha a)^{-(T+1)/2} \times \exp N \frac{(a-1)^2\|c^{\mu^N}\|^2}{2(1 - \alpha a)\sigma^2}.$$

Since

$$\Gamma_1(\mu^N) + \beta_1 \leq \beta_1,$$

we have

$$\Gamma_1(\mu^N) + \beta \leq \beta.$$

Putting all this together we obtain

$$\int_{\mathcal{T}^N} \exp(a\phi^N(x))\, P^{\otimes N}(dx) \leq \exp\left(Nc + aN\beta\right), \tag{69}$$

where

$$c = -\frac{T+1}{2}\log(1-a\alpha) + \frac{a(a-1)(1-\alpha)}{2\sigma^2(1-\alpha a)}\|c^{\mu^N}\|^2 > 0.$$

□

We may now conclude the proof of the proposition.

Proof. We take $M \to \infty$ and apply the dominated convergence theorem to (67), noting that ϕ^N_M grows to ϕ^N. We thus find that

$$a\int_{\mathcal{T}^N} \phi^N(x)\mu^N(dx) \leq \log\int_{\mathcal{T}^N} \exp\left(a\phi^N(x)\right) P^{\otimes N}(dx) + I^{(2)}(\mu^N, P^{\otimes N}).$$

This and (69) imply that, for all N,

$$aN\left(\Gamma(\mu^N) + \beta\right) \leq Nc + aN\beta + I^{(2)}(\mu^N, P^{\otimes N}),$$

as required. We divide both sides by aN and let $N \to \infty$ to obtain the required result. □

Proposition 31. *The family $\{\Pi^N\}$ is exponentially tight.*

Proof. Let $B \in \mathcal{B}(\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}}))$. We have

$$\Pi^N(B) = \int_{(\hat{\mu}^N)^{-1}(B)} \exp N\Gamma(\hat{\mu}^N(x))\, P^{\otimes N}(dx).$$

Through Hölder's Inequality, we find that for any $a > 1$ such that $1 - a\alpha > 0$:

$$\Pi^N(B) \leq R^N(B)^{(1-\frac{1}{a})}\left(\int_{(\hat{\mu}^N)^{-1}(B)} \exp\left(aN\Gamma(\hat{\mu}^N(x))\right) P^{\otimes N}(dx)\right)^{\frac{1}{a}},$$

Now it may be observed that $N\Gamma(\hat{\mu}^N(x)) = \phi^N(x) - N\beta$, where Φ^N is defined by (66). It therefore follows from lemma 30 that

$$\Pi^N(B) \leq R^N(B)^{(1-\frac{1}{a})}\exp\left(\frac{Nc}{a}\right). \tag{70}$$

By the exponential tightness of $\{R^N\}$ (as proved in lemma 26), for each $L > 0$, there exists a compact set K_L such that

$$\overline{\lim_{N\to\infty}} N^{-1} \log(R^N(K_L^c)) \leq -L.$$

It may be seen that if we choose

$$B = K^c_{\frac{a}{a-1}(L+\frac{c}{a})}$$

then we obtain from (70) that

$$\overline{\lim_{N\to\infty}} N^{-1} \log \Pi^N(B) \leq -L$$

as required. □

5.3 Upper Bound on the Compact Sets

In this section we obtain an upper bound on the compact sets, i.e. the first half of theorem 2 for F is compact. Our method is to obtain an LDP for a simplified Gaussian system (with fixed A^ν and c^ν), and then prove that this converges to the required bound as $\nu \to \mu$.

5.3.1 An LDP for a Gaussian measure

We linearise Γ in the following manner. Fix $\nu \in \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ and assume for the moment that $\mu \in \mathcal{E}_2$. Let

$$\Gamma_2^\nu(\mu^N) = \frac{1}{2\sigma^2} \int_{\mathcal{S}^N} \left(\sum_{k=-n}^{n} {}^t(v^0 - c^\nu) A^{\nu,\,N,\,k} (v^k - c^\nu) + 2 \left\langle c^\nu, v^0 \right\rangle - \|c^\nu\|^2 \right) \underline{\underline{\mu}}^N(dv), \quad (71)$$

where $A^{\nu,N,k}$, $k = -n, \cdots, n$ is the kth $(T+1) \times (T+1)$ block of the $N(T+1) \times N(T+1)$ symmetric block circulant matrix

$$K^{\nu,N}(\sigma^2 \mathrm{Id}_{N(T+1)} + K^{\nu,N})^{-1},$$

and $K^{\nu,N}$ is the $N(T+1) \times N(T+1)$ covariance matrix of the Gaussian process $(G^{\nu,j})_{j=-n,\cdots,n}$ defined in section 4.1.2.

Let us also define

$$\Gamma_1^N(\nu) = -\frac{1}{2N} \log\det\left(\mathrm{Id}_{N(T+1)} + \frac{1}{\sigma^2} K^{\nu,N}\right),$$

and let

$$\Gamma^\nu(\mu^N) = \Gamma_1^N(\nu) + \Gamma_2^\nu(\mu^N).$$

We let $\Gamma_a^\nu(\mu) = \lim_{N\to\infty} \Gamma_a^\nu(\mu^N)$, (for $a = 1$ or 2) and find, using the seecond identity in proposition 20, that

$$\Gamma_2^\nu(\mu) = \frac{1}{2\sigma^2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{A}^\nu(\omega) : \tilde{v}^\mu(d\omega) \right.$$
$$\left. -2\,{}^t c^\nu \tilde{A}^\nu(0)\bar{v}^\mu + {}^t c^\nu \tilde{A}^\nu(0) c^\nu + 2\langle c^\nu, \bar{v}^\mu\rangle - \|c^\nu\|^2\right), \quad (72)$$

where $\bar{v}^\mu = \mathbb{E}^{\underline{\underline{\mu}}}[v^0]$, and $\tilde{v}^\mu$ is the spectral measure of the process $(\Psi \circ f^{-1}(x^k))_{k\in\mathbb{Z}}$ given in (30). The spectral measure exists as long as $\mu \in \mathcal{E}_2$, in which case the above is finite. We recall that : denotes double contraction on the indices.

Similarly to lemma 18, we find that

$$\lim_{N\to\infty} \Gamma_1^N(\nu) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\log\det\left(\mathrm{Id}_{T+1} + \frac{1}{\sigma^2}\tilde{K}^\nu(\omega)\right)\right) d\omega = \Gamma_1(\nu). \quad (73)$$

For $\mu \in \mathcal{E}_2$, we define $H^\nu(\mu) = I^{(3)}(\mu, P^{\mathbb{Z}}) - \Gamma^\nu(\mu)$; for $\mu \notin \mathcal{E}_2$, we define $\Gamma_2^\nu(\mu) = \Gamma^\nu(\mu) = \infty$ and $H^\nu(\mu) = \infty$. In fact it will be seen that H^ν is the rate function for the Gaussian Stationary Process Q^ν to be defined below.

We define the following measure over $\mathcal{S}^N$. For $B \in \mathcal{B}(\mathcal{S}^N)$,

$$\underline{\underline{Q}}^{\nu,N}(B) = \int_B \exp\left(N\Gamma^\nu(\underline{\underline{\hat{\mu}}}^N(v))\right) \underline{\underline{P}}^{\otimes N}(dv). \quad (74)$$

This defines a law $Q^{\nu,N}$ over $\mathcal{T}^N$ according to the correspondence in definition 2. We find that

$$\underline{\underline{Q}}^{\nu,N}(B) = \left(\det\left(\mathrm{Id}_{N(T+1)} + \frac{1}{\sigma^2}K^{\nu,N}\right)\right)^{-\frac{1}{2}} \times$$
$$\int_B \exp\frac{1}{2\sigma^2}\left(\sum_{j,k=-n}^{n} {}^t(v^j - c^\nu)A^{\nu,N,k}(v^{k+j} - c^\nu) + \right.$$
$$\left. 2\sum_{j=-n}^{n}\langle c^\nu, v^j\rangle - N\|c^\nu\|^2\right)\underline{\underline{P}}^{\otimes N}(dv). \quad (75)$$

We note $c^{\nu,N}$ the $N(T+1)$-dimensional vector obtained by concatenating N times the vector c^ν. We also have that

$$\frac{1}{\sigma^2}(\mathrm{Id}_{N(T+1)} - A^{\nu,N}) = (\sigma^2 \mathrm{Id}_{N(T+1)} + K^{\nu,N})^{-1}.$$

Thus, through proposition 1, we find that

$$\underline{\underline{Q}}^{\nu,N}(B) = (2\pi)^{-\frac{N(T+1)}{2}} \left(\det \left(\frac{1}{\sigma^2}(\mathrm{Id}_{N(T+1)} - A^{\nu,N}) \right)^{-1} \right)^{-\frac{1}{2}}$$

$$\int_B \exp -\frac{1}{2\sigma^2}{}^t \left(v - c^{\nu,N}\right) \left(\mathrm{Id}_{N(T+1)} - A^{\nu,N}\right) \left(v - c^{\nu,N}\right) \prod_{j=-n}^{n} \prod_{t=0}^{T} dv_t^j. \quad (76)$$

It is seen that $\underline{\underline{Q}}^{\nu,N}$ is an $N(T+1)$-dimensional Gaussian measure with mean $c^{\nu,N}$, inverse covariance matrix $\frac{1}{\sigma^2}(\mathrm{Id}_{N(T+1)} - A^{\nu,N})$, and covariance matrix $\sigma^2 \mathrm{Id}_{N(T+1)} + K^{\nu,N}$. Hence $\underline{\underline{Q}}^{\nu,N}$ is in $\mathcal{M}^+_{1,s}(\mathcal{S}^N)$.

We may thus define the measure $\underline{\underline{Q}}^\nu$ of a stationary Gaussian process over the variables $\{v_s^j\}_{j\in\mathbb{Z},s=0,\cdots,T}$, with N-dimensional marginals given by (76). The corresponding infinite dimensional Gaussian measure $\underline{\underline{Q}}^\nu$ on $\mathcal{S}^{\mathbb{Z}}$ has covariance operator $\sigma^2\mathrm{Id} + K^\nu$ and mean c^ν. It may be observed that the spectral density of the covariance is $\sigma^2 \mathrm{Id}_{T+1} + \tilde{K}^\nu$.

Let $\underline{\underline{\Pi}}^{\nu,N}$ be the image law of $\underline{\underline{Q}}^\nu$ under $\underline{\underline{\hat{\mu}}}^N$, i.e. for $B \in \mathcal{B}(\mathcal{M}^+_{1,s}(\mathcal{S}^{\mathbb{Z}}))$,

$$\underline{\underline{\Pi}}^{\nu,N}(B) = \underline{\underline{Q}}^\nu \left(\underline{\underline{\hat{\mu}}}^N \in B \right).$$

Lemma 32. *The image law $\underline{\underline{\Pi}}^{\nu,N}$ satisfies a strong LDP (in the manner of proposition 2) with good rate function*

$$\underline{\underline{H}}^\nu(\underline{\underline{\mu}}) = I^{(3)}(\underline{\underline{\mu}}, \underline{\underline{P}}^{\mathbb{Z}}) - \Gamma^\nu(\mu), \quad (77)$$

where $I^{(3)} : \mathcal{M}^+_{1,s}(\mathcal{S}^{\mathbb{Z}}) \to \mathbb{R} \cup \infty$ is defined analogously to (63).

This result is deduced from [2, 21], see appendix A. For $B \in \mathcal{B}(\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}}))$, we define the image law

$$\Pi^{\nu,N}(B) = Q^\nu(\hat{\mu}^N \in B) = \underline{\underline{Q}}^\nu(\underline{\underline{\mu}}^N \in \Psi \circ f^{-1}(B)).$$

It follows from the contraction principle that if we write $H^\nu(\mu) := \underline{\underline{H}}^\nu(\underline{\underline{\mu}})$, then

Corollary 33. *The image law* $\Pi^{\nu,N}$ *satisfies a strong LDP with good rate function*

$$H^{\nu}(\mu) = I^{(3)}(\mu, P^{\mathbb{Z}}) - \Gamma^{\nu}(\mu). \tag{78}$$

In particular, we note that $I^{(3)}(\underline{\underline{\mu}}, \underline{\underline{P}}^{\mathbb{Z}}) = I^{(3)}(\mu, P^{\mathbb{Z}})$.

5.3.2 An upper bound for Π^N over compact sets

In this section we derive an upper bound for Π^N over compact sets using the LDP of the previous section. Before we do this, we require some lemmas governing the 'distance' between Γ^{ν} and Γ. Let

$$C_N^{\nu} = \sup_{M \geq N, (2|l|+1) \leq M} \{\|\tilde{A}^{\nu^M,l} - \tilde{A}^{\nu,M,l}\|, \|\tilde{K}^{\nu^M,l} - \tilde{K}^{\nu,M,l}\|\}, \tag{79}$$

where we have taken the operator norm. Recall that $\tilde{K}^{\nu,M}$ and $\tilde{A}^{\nu,M}$ are defined in Section 5.3.1.

Lemma 34. *For all* $\nu \in \mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$, C_N^{ν} *is finite and*

$$C_N^{\nu} \to 0 \text{ as } N \to \infty.$$

Proof. We recall from proposition 29 that $\tilde{K}_{st}^{\nu^M}(\omega)$ converges uniformly (in ω) to $\tilde{K}_{st}^{\nu}(\omega)$. The same holds for $\tilde{K}_{st}^{\nu,M,l}$, because this represents the partial summation of an absolutely converging Fourier Series. That is, for fixed $\omega = 2\pi l_M/M$, $\tilde{K}_{st}^{\nu,M,l_M} \to \tilde{K}_{st}^{\nu}(\omega)$ as $M \to \infty$. The result then follows from the equivalence of matrix norms. The proof for $\tilde{A}^{\nu}$ is analogous. ☐

Lemma 35. *There exists a constant* C_0 *such that for all* ν *in* $\mathcal{M}_{1,s}^{+}(\mathcal{T}^{\mathbb{Z}})$, *all* $\epsilon > 0$ *and all* $\mu \in V_{\epsilon}(\nu) \cap \mathcal{E}_2$,

$$\left|\Gamma(\mu^N) - \Gamma^{\nu}(\mu^N)\right| \leq C_0(C_N^{\nu} + \epsilon)(1 + \mathbb{E}^{\underline{\underline{\mu}}}[\|v^0\|^2]).$$

Here $V_{\varepsilon}(\nu)$ *is the open neighbourhood defined in proposition 17, and* $\underline{\underline{\mu}}$ *is given in definition 2.*

Proof. We firstly bound Γ_1.

$$\begin{aligned}
&\left|\Gamma_1(\mu^N) - \Gamma_1^N(\nu)\right| \leq \\
&\quad \frac{1}{2N}\sum_{l=-n}^{n}\left|\log\det\left(\mathrm{Id}_{T+1} + \sigma^{-2}\tilde{K}^{\mu^N,l}\right) - \log\det\left(\mathrm{Id}_{T+1} + \sigma^{-2}\tilde{K}^{\nu^N,l}\right)\right| \\
&\quad + \frac{1}{2N}\sum_{l=-n}^{n}\left|\log\det\left(\mathrm{Id}_{T+1} + \sigma^{-2}\tilde{K}^{\nu^N,l}\right) - \log\det\left(\mathrm{Id}_{T+1} + \sigma^{-2}\tilde{K}^{\nu,N,l}\right)\right|.
\end{aligned} \tag{80}$$

It thus follows from proposition 17 and lemma 34 that

$$\left|\Gamma_1(\mu^N) - \Gamma_1^N(\nu)\right| \leq C_0^*(C_N^\nu + \epsilon),$$

for some constant C_0^* which is independent of ν and N.

It remains for us to bound Γ_2. The proof uses a slightly modified version of the spectral representation used in the proof of lemma 30. We perform the bijective affine change of variables in $\mathcal{S}^N$

$$h : (v^{-n}, \cdots, v^n) \to (y^{-n}, \cdots, y^n),$$

where, for $s \in [0, T]$,

$$y_s^k = \left\{\begin{array}{ll} \sqrt{2}\mathrm{Re}(\tilde{v}_s^{-k}) & k = -1, \cdots, -n \\ \tilde{v}_s^0 & k = 0 \\ \sqrt{2}\mathrm{Im}(\tilde{v}_s^k) & k = 1, \cdots, n \end{array}\right. . \tag{81}$$

$(\tilde{v}^k)$, $k = -n, \cdots, n$ is the discrete Fourier transform of the sequence (v^k).

This allows us to write the integrand of $\Gamma_2^\nu(\mu^N)$ (up to the factor $1/2\sigma^2$) as

$$\frac{1}{N^2}\sum_{l=-n}^{n} {}^t y^l \tilde{A}^{\nu,N,l} y^l + \frac{2}{N}\langle c^\nu - \tilde{A}^{\nu,N,0}c^\nu, y^0\rangle + {}^t c^\nu \tilde{A}^{\nu,N,0} c^\nu - \|c^\nu\|^2,$$

Similarly we write the integrand for $\Gamma_2(\mu^N)$

$$\frac{1}{N^2}\sum_{l=-n}^{n} {}^t y^l \tilde{A}^{\mu^N,l} y^l + \frac{2}{N}\langle c^\mu - \tilde{A}^{\mu^N,0}c^\mu, y^0\rangle + {}^t c^\mu \tilde{A}^{\mu^N,0} c^\mu - \|c^\mu\|^2.$$

Hence we have

$$\left|\Gamma_2(\mu^N) - \Gamma_2^\nu(\mu^N)\right| \leq$$
$$\frac{1}{2\sigma^2}\int_{\mathcal{S}^N}\left(\frac{1}{N^2}\sum_{l=-n}^{n}\|\tilde{A}^{\mu^N,l}-\tilde{A}^{\nu,N,l}\|\|y^l\|^2+\frac{2}{N}\|d_{\nu,\mu}\|\|y^0\|+|e_{\nu,\mu}|\right)\underline{\underline{\mu}}^N\circ h^{-1}(dy),$$

where $d_{\nu,\mu} = c^\mu - c^\nu + \tilde{A}^{\nu,N,0}c^\nu - \tilde{A}^{\mu^N,0}c^\mu$and $e_{\nu,\mu} = {}^t c^\mu \tilde{A}^{\mu^N,0}c^\mu - \|c^\mu\|^2 - {}^t c^\nu \tilde{A}^{\nu,N,0}c^\nu + \|c^\nu\|^2$. Here $\|\tilde{A}^{\mu^N,l}-\tilde{A}^{\nu,N,l}\|$ is the operator norm but $\|d_{\nu,\mu}\|$ is the vector norm. We may bound the coefficients through the following identities. It was proved in proposition 17 that for all $0 \leq s,t \leq T$, $\left|\tilde{A}^{\mu^N,l}_{st} - \tilde{A}^{\nu^N,l}_{st}\right| \leq \epsilon$ and $|c^\nu_s - c^\mu_s| \leq \varepsilon$. We may thus infer that $\|\tilde{A}^{\mu^N,l} - \tilde{A}^{\nu^N,l}\| \leq (T+1)\varepsilon$ and $\|c^\nu - c^\mu\| \leq (T+1)\varepsilon$. Furthermore, we have from lemma 34 that $\|\tilde{A}^{\nu^N,l} - \tilde{A}^{\nu,N,l}\| \leq C^\nu_N$, and $\|c^\nu\|$ is bounded by $T\bar{J}^2$ for all ν.

We thus observe that

$$\|\tilde{A}^{\mu^N,l} - \tilde{A}^{\nu,N,l}\| \leq \|\tilde{A}^{\mu^N,l} - \tilde{A}^{\nu^N,l}\| + \|\tilde{A}^{\nu^N,l} - \tilde{A}^{\nu,N,l}\| \quad l = -n,\cdots,n,$$
$$\|d_{\nu,\mu}\| \leq \|c^\nu - c^\mu\| + \|\tilde{A}^{\nu^N,0}c^\nu - \tilde{A}^{\mu^N,0}c^\mu\| + \|\left(\tilde{A}^{\nu,N,0} - \tilde{A}^{\nu^N,0}\right)c^\nu\|,$$
$$|e_{\nu,\mu}| \leq |{}^t c^\mu \tilde{A}^{\mu^N,0}c^\mu - {}^t c^\nu \tilde{A}^{\nu^N,0}c^\nu| + |{}^t c^\nu\left(\tilde{A}^{\nu^N,0} - \tilde{A}^{\nu,N,0}\right)c^\nu|.$$

It is evident from the above considerations that each of the above terms is bounded by $C^*(C^\nu_N + \epsilon)$ for some constant C^*. The lemma now follows after consideration of the fact that $\int_{\mathcal{S}^{\mathbb{Z}}}\|v^k\|^2\underline{\underline{\mu}}(dv) = \mathbb{E}^{\underline{\underline{\mu}}}[\|v^0\|^2]$, $\|y^0\|^2 \leq N\sum_{k=-n}^{n}\|v^k\|^2$ and, because of the properties of the discrete Fourier transform

$$\sum_{l=-n}^{n}\|y^l\|^2 = N\sum_{k=-n}^{n}\|v^k\|^2. \tag{82}$$

□

We are now ready to begin the proof of the upper bound on compact sets.

Proposition 36. *Let K be a compact subset of $\mathcal{M}_{1,s}(\mathcal{T}^{\mathbb{Z}})$. Then* $\overline{\lim}_{N\to\infty} N^{-1}\log(\Pi^N(K)) \leq -\inf_K H$.

Proof. Fix $\varepsilon > 0$. Let $V_\varepsilon(\nu)$ be the open neighbourhood of ν defined in proposition 17, and let $\bar{V}_\varepsilon(\nu)$ be its closure. Since K is compact and $\{V_\varepsilon(\nu)\}_{\nu\in K}$ is

an open cover, there exists an r and $\{\nu_i\}_{i=1}^r$ such that $K \subset \bigcup_{i=1}^r V_\varepsilon(\nu_i)$. We find that

$$\overline{\lim_{N\to\infty}} N^{-1} \log \left(\Pi^N \left(\bigcup_{i=1}^r V_\varepsilon(\nu_i) \cap K \right) \right) \\ \leq \sup_{1\leq i\leq r} \overline{\lim_{N\to\infty}} N^{-1} \log \left(\Pi^N \left(\bar{V}_\varepsilon(\nu_i) \cap K \right) \right).$$

It follows from the fact that $\hat{\mu}^N \in \mathcal{E}_2$, lemma 35 and the definition of Π^N that

$$\Pi^N(\bar{V}_\varepsilon(\nu_i) \cap K) \leq \int_{\hat{\mu}^N(x)\in\bar{V}_\varepsilon(\nu_i)\cap K} \exp \Bigg(N\Gamma^{\nu_i}(\hat{\mu}^N(x)) + \\ NC_0(\varepsilon + C_N^{\nu_i}) \left(1 + \frac{1}{N} \sum_{j=-n}^{n} \|v^j\|^2 \right) \Bigg) P^{\otimes N}(dx), \quad (83)$$

where $v^j = \Psi(f^{-1}(x^j))$. From the definition of $Q^{\nu,N}$ in (74) and Hölder's Inequality, for p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\Pi^N(\bar{V}_\varepsilon(\nu_i) \cap K) \leq \left(Q^{\nu_i,N}(\hat{\mu}^N(x) \in \bar{V}_\varepsilon(\nu_i) \cap K) \right)^{\frac{1}{p}} D^{\frac{1}{q}}, \quad (84)$$

where

$$D = \int_{\hat{\mu}^N(x)\in\bar{V}_\varepsilon(\nu_i)\cap K} \exp \left(qNC_0(\varepsilon + C_N^{\nu_i}) \left(1 + \frac{1}{N} \sum_{j=-n}^{n} \|v^j\|^2 \right) \right) Q^{\nu_i,N}(dx) \\ = \exp qNC_0(\varepsilon + C_N^{\nu_i}) \times \\ \int_{\underline{\underline{\hat{\mu}}}^N(v)\in\Psi\circ f^{-1}(\bar{V}_\varepsilon(\nu_i)\cap K)} \exp \left(qC_0(\varepsilon + C_N^{\nu_i}) \left(\sum_{j=-n}^{n} \|v^j\|^2 \right) \right) \underline{\underline{Q}}^{\nu_i,N}(dv).$$

We note from lemma 14 that the eigenvalues of the covariance of $\underline{\underline{Q}}^{\nu_i,N}$ are upperbounded by $\sigma^2 + \rho_K$. Thus for this integral to converge it is sufficient that

$$qC_0(\epsilon + C_N^{\nu_i}) \leq \frac{1}{2(\sigma^2 + \rho_K)}. \quad (85)$$

This condition will always be satisfied for sufficiently small ϵ and sufficiently large N (since $C_N^{\nu_i} \to 0$ as $N \to \infty$). By corollary 33,

$$\overline{\lim_{N\to\infty}} N^{-1} \log \left(Q^{\nu_i,N}(\hat{\mu}^N(x) \in \bar{V}_\varepsilon(\nu_i) \cap K) \right) \leq - \inf_{\mu\in\bar{V}_\varepsilon(\nu_i)\cap K} H^{\nu_i}(\mu). \quad (86)$$

We know that $\underline{\underline{Q}}^{\nu_i,N}$ is Stationary Gaussian with mean c^{ν_i} and covariance $\sigma^2 \mathrm{Id}_{N(T+1)} + K^{\nu_i,N}$. We apply lemma 6 to find

$$\int_{\underline{\underline{\hat{\mu}}}^N(v)\in\Psi\circ f^{-1}(\bar{V}_\varepsilon(\nu_i)\cap K)} \exp qC_0(\varepsilon + C_N^{\nu_i}) \left(\sum_{j=-n}^{n} \|v^j\|^2\right) \underline{\underline{Q}}^{\nu_i,N}(dv) \leq$$
$$\left(\det\left((1-2qC_0(\varepsilon+C_N^{\nu_i})\sigma^2)\mathrm{Id}_{N(T+1)} - 2qC_0(\varepsilon+C_N^{\nu_i})K^{\nu_i,N}\right)\right)^{-\frac{1}{2}} \times$$
$$\exp\left(2C_0^2 q^2((\varepsilon+C_N^{\nu_i})^2)^t(1_{N(T+1)}c^{\nu_i})B(1_{N(T+1)}c^{\nu_i}) + NqC_0(\varepsilon+C_N^{\nu_i})\|c^{\nu_i}\|^2\right)$$

where $1_{N(T+1)}$ is the $N(T+1)\times(T+1)$ block matrix with each block $\mathrm{Id}_{\mathrm{T+1}}$ and

$$B = (\sigma^2\mathrm{Id}_{N(T+1)} + K^{\nu_i,N})((1-2C_0q(\varepsilon+C_N^{\nu_i})\sigma^2)\mathrm{Id}_{N(T+1)} - 2C_0q(\varepsilon+C_N^{\nu_i})K^{\nu_i,N})^{-1}$$

is a symmetric block circulant matrix.

We note B^k, $k=-n,\cdots,n$ its $T\times T$ blocks. We have

$${}^t(1_{N(T+1)}c^{\nu_i})B(1_{N(T+1)}c^{\nu_i}) = N\,{}^tc^{\nu_i}\left(\sum_{k=-n}^{n} B^k\right)c^{\nu_i} = N\,{}^tc^{\nu_i}\tilde{B}^0 c^{\nu_i},$$

where $\tilde{B}^0$ is the 0th component of the spectral representation of the sequence $(B^k)_{k=-n,\cdots,n}$. Let v_m be the largest eigenvalue of B. Since (by lemma 5) the eigenvalues of $\tilde{B}^0$ are a subset of the eigenvalues of B, we have

$${}^t(1_{N(T+1)}c^{\nu_i})B(1_{N(T+1)}c^{\nu_i}) \leq N v_m \|c^{\nu_i}\|^2.$$

From the definition of B and through lemma 14 we have

$$v_m \leq \frac{\sigma^2+\rho_K}{1-2C_0q(\varepsilon+C_N^{\nu_i})(\sigma^2+\rho_K)}.$$

Hence we have, since $\|c^{\nu_i}\|^2 \leq T\bar{J}^2$

$$\exp\left(2C_0^2(q^2(\varepsilon+C_N^{\nu_i})^{2t}(1_{N(T+1)}c^{\nu_i})B(1_{N(T+1)}c^{\nu_i})\right) \leq$$
$$\exp\left(NT\times\frac{2C_0^2q^2(\varepsilon+C_N^{\nu_i})^2(\sigma^2+\rho_K)\bar{J}^2}{1-2C_0q(\varepsilon+C_N^{\nu_i})(\sigma^2+\rho_K)}\right).$$

Since the determinant is the product of the eigenvalues, we similarly find that

$$\left(\det\left((1-2C_0q(\varepsilon+C_N^{\nu_i})\sigma^2)\mathrm{Id}_{N(T+1)}-2C_0q(\varepsilon+C_N^{\nu_i})K^{\nu_i,N}\right)\right)^{-\frac{1}{2}}\leq \left(1-2C_0q(\varepsilon+C_N^{\nu_i})(\sigma^2+\rho_K)\right)^{-\frac{N(T+1)}{2}}.$$

Upon collecting the above inequalities, and noting that $\|c^\nu\|^2\leq (T+1)\bar{J}^2$, we find that

$$D\leq \exp(Ns_N^{\nu_i}(q,\epsilon)), \tag{87}$$

where

$$s_N^{\nu_i}(q,\varepsilon)=(T+1)\left(-\frac{1}{2}\log\left(1-2C_0q(\varepsilon+C_N^{\nu_i})(\sigma^2+\rho_K)\right)\right. \\ \left.+\frac{2C_0^2q^2(\varepsilon+C_N^{\nu_i})^2(\sigma^2+\rho_K)\bar{J}^2}{1-2C_0q(\varepsilon+C_N^{\nu_i})(\sigma^2+\rho_K)}+qC_0(\varepsilon+C_N^{\nu_i})\left(\frac{1}{T+1}+\bar{J}^2\right)\right).$$

We let $s(q,\varepsilon)=\overline{\lim}_{N\to\infty}s_N^{\nu_i}(q,\varepsilon)$, and find through lemma 34 that

$$s(q,\varepsilon)=(T+1)\left(-\frac{1}{2}\log\left(1-2C_0q\varepsilon(\sigma^2+\rho_K)\right)\right. \\ \left.+\frac{2C_0^2q^2\varepsilon^2(\sigma^2+\rho_K)\bar{J}^2}{1-2C_0q\varepsilon(\sigma^2+\rho_K)}+qC_0\varepsilon\left(\frac{1}{T+1}+\bar{J}^2\right)\right).$$

Notice that $s(q,\varepsilon)$ is independent of ν_i and that $s(q,\varepsilon)\to 0$ as $\varepsilon\to 0$. Using (84), (86) and (87) we thus find that

$$\overline{\lim_{N\to\infty}}N^{-1}\log(\Pi^N(K))\leq \sup_{1\leq i\leq r}-\frac{1}{p}\inf_{\mu\in K\cap\bar{V}_\varepsilon(\nu_i)}H^{\nu_i}(\mu)-\frac{1}{q}s(q,\varepsilon).$$

Recall that $H^\nu(\mu)=\infty$ for all $\mu\notin\mathcal{E}_2$. Thus if $K\cap\mathcal{E}_2=\emptyset$, we may infer that $\overline{\lim}_{N\to\infty}N^{-1}\log(\Pi^N(K))=-\infty$ and the proposition is evident. Thus we may assume without loss of generality that $\inf_{\mu\in K}H^{\nu_i}(\mu)=\inf_{\mu\in K\cap\mathcal{E}_2}H^{\nu_i}(\mu)$. Furthermore it follows from proposition 37 (below) that there exists a constant C_I such that for all $\mu\in\bar{V}_\epsilon(\nu_i)\cap\mathcal{E}_2$,

$$H^{\nu_i}(\mu)\geq I^{(3)}(\mu,P^{\mathbb{Z}})-\Gamma(\mu)-C_I\epsilon(1+I^{(3)}(\mu,P^{\mathbb{Z}})).$$

We thus find that

$$\overline{\lim_{N\to\infty}} N^{-1}\log(\Pi^N(K)) \leq \\ -\frac{1}{p}\inf_{K\cap\mathcal{E}_2}\left(I^{(3)}(\mu,P^{\mathbb{Z}})(1-C_I\varepsilon)-\Gamma(\mu)\right)-\frac{s(q,\varepsilon)}{q}+\frac{\varepsilon}{p}C_I,$$

We take $\varepsilon\to 0$ and find, through the use of proposition 29, that

$$\overline{\lim_{N\to\infty}} N^{-1}\log(\Pi^N(K)) \leq -\frac{1}{p}\inf_{K}\left(I^{(3)}(\mu,P^{\mathbb{Z}})-\Gamma(\mu)\right).$$

The proof may thus be completed by taking $p\to 1$. □

Proposition 37. *There exists a positive constant C_I such that, for all ν in $\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})\cap\mathcal{E}_2$, all $\varepsilon>0$ and all $\mu\in\bar{V}_\varepsilon(\nu)\cap\mathcal{E}_2$ (where $\bar{V}_\varepsilon(\nu)$ is the neighbourhood defined in proposition 17),*

$$|\Gamma^\nu(\mu)-\Gamma^\mu(\mu)|\leq C_I\varepsilon\left(1+I^{(3)}(\mu,P^{\mathbb{Z}})\right). \tag{88}$$

The proof is very similar to that of proposition 29, and we have therefore left it in the Appendix.

5.4 H is a good rate function

Lemma 38. *$H(\mu)$ is lower-semi-continuous.*

Proof. Fix μ and let $(\mu_m)_{m\geq 0}$ converge weakly to μ as $m\to\infty$. We let (μ_{p_m}) be a subset such that $\underline{\lim}_{m\to\infty} H(\mu_m)=\lim_{m\to\infty}H(\mu_{p_m})$. Suppose firstly that

$$\overline{\lim_{m\to\infty}} I^{(3)}\left(\mu_{p_m},P^{\mathbb{Z}}\right)=\infty. \tag{89}$$

From proposition 29 we have that, if $\mu_{p_m}\in\mathcal{E}_2$, then $H(\mu_{p_m})\geq\left(1-\frac{1}{a}\right)I^{(3)}(\mu_{p_m})-\frac{c}{a}$, where $a>1$ and $c>0$ are constants. Otherwise, if $\mu_{p_m}\notin\mathcal{E}_2$ then (through lemma 27) $H(\mu_{p_m})=\infty$. In either case, we find that $\overline{\lim}_{m\to\infty} H(\mu_{p_m})=\lim_{m\to\infty}H(\mu_{p_m})=\infty$, so that in this instance H is lower-semicontinuous at μ.

In the second instance, we assume that (89) does not hold, so that there exists an M such that for all $m\geq M$, $\{I^{(3)}\left(\mu_{p_m},P^{\mathbb{Z}}\right)\}$ is upperbounded (and

by lemma 27, $\mu_{p_m} \in \mathcal{E}_2$). We then find that

$$\begin{aligned}\underline{\lim}_{m\to\infty} H\left(\mu_{p_m}\right) &= \underline{\lim}_{m\to\infty} \left(I^{(3)}(\mu_{p_m}, P^{\mathbb{Z}}) - \Gamma(\mu_{p_m})\right) \\ &\geq \underline{\lim}_{m\to\infty} H^{\mu}(\mu_{p_m}) + \underline{\lim}_{m\to\infty} (\Gamma^{\mu} - \Gamma)(\mu_{p_m}).\end{aligned}$$

Recall that $\Gamma(\mu_{p_m}) = \Gamma^{\mu_{p_m}}(\mu_{p_m})$. It follows from proposition 37 and the boundedness of $I^{(3)}\left(\mu_{p_m}\right)$ that the second term is zero. However H^{μ} is lower-semi-continuous [2], which allows us to conclude that $\underline{\lim}_{m\to\infty} H^{\mu}(\mu_{p_m}) \geq H^{\mu}(\mu) = H(\mu)$ as required. $\square$

Because $\{\Pi^N\}$ is exponentially tight and satisfies the weak LDP with rate function $H(\mu)$, the following corollary is immediate [20, Lemma 2.15].

Corollary 39. *$H(\mu)$ is a good rate function, i.e. the sets $\{\mu : H(\mu) \leq \delta\}$ are compact for all $\delta \in \mathbb{R}^+$.*

6 The unique minimum of the rate function

We first prove that there exists a unique minimum μ_e of the rate function, before proving that Π^N converges weakly to δ_{μ_e}. We finish by providing explicit equations for μ_e which would facilitate its numerical simulation.

Lemma 40. *For $\mu, \nu \in \mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$, $H^{\mu}(\nu) = 0$ if and only if $\nu = Q^{\mu}$.*

Proof. Using the correspondences in section 5.3.1, it suffices to prove that $\underline{\underline{H}}^{\mu}(\underline{\underline{\nu}}) = 0$ if and only if $\underline{\underline{\nu}} = \underline{\underline{Q}}^{\mu}$. Let $(\mathcal{K}^{\mu,k})_{k\in\mathbb{Z}}$ be the Fourier Coefficients of $\left(\sigma^2\mathrm{Id} + \tilde{\mathrm{K}}^{\mu}\right)^{-\frac{1}{2}}$, i.e.

$$\mathcal{K}^{\mu,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(ik\omega\right)\left(\sigma^2\mathrm{Id}_{\mathrm{T}+1} + \tilde{\mathrm{K}}^{\mu}(\omega)\right)^{-\frac{1}{2}} d\omega.$$

This is well-defined because $\sigma^2\mathrm{Id}_{\mathrm{T}+1} + \tilde{\mathrm{K}}^{\mu}(\omega)$ is symmetric and its eigenvalues are strictly positive. Let $\tau^{\mu} : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ be the map $v \to (\tau^{\mu}(v))^k = \sum_{l=-\infty}^{\infty} \mathcal{K}^{\mu,l}(v^{k-l} - c^{\mu})$, and let $\underline{\underline{P}}_0 \simeq \mathcal{N}_{T+1}(\mathbf{0}_{T+1}, \mathrm{Id}_{T+1})$.

The result in [21] stipulates that

$$\underline{\underline{H}}^{\mu}(\underline{\underline{\nu}}) = I^{(3)}\left(\underline{\underline{\nu}} \circ (\tau^{\mu})^{-1}, \underline{\underline{P}}_0^{\mathbb{Z}}\right).$$

In turn a contraction principle [23] dictates that

$$I^{(3)}\left(\underline{\underline{\nu}}\circ(\tau^\mu)^{-1},\underline{\underline{P}}_0^{\mathbb{Z}}\right)\geq 0,$$

with equality if and only if $\underline{\underline{\nu}}\circ(\tau^\mu)^{-1}=\underline{\underline{P}}_0^{\mathbb{Z}}$. We note that $\underline{\underline{P}}_0^{\mathbb{Z}}=\underline{\underline{Q}}^\mu\circ(\tau^\mu)^{-1}$, and that if $H^\mu(\nu)=I^{(3)}\left(\underline{\underline{\nu}}\circ(\tau^\mu)^{-1},\underline{\underline{P}}_0^{\mathbb{Z}}\right)<\infty$ then $\underline{\underline{\nu}}$ is absolutely continuous with respect to $\underline{\underline{P}}_0^{\mathbb{Z}}$ (as noted in Section 5.3.1). The lemma now follows from the fact that τ^μ is one-to-one on the set of all measures absolutely continuous with respect to $\underline{\underline{P}}_0^{\mathbb{Z}}$. □

Proposition 41. *There is a unique distribution $\mu_e\in\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ which minimises H. This distribution satisfies $H(\mu_e)=0$.*

Proof. By the previous lemma, it suffices to prove that there is a unique μ_e such that

$$Q^{\mu_e}=\mu_e. \tag{90}$$

Let $\mathcal{F}_t$ be the σ-algebra over $\mathcal{T}^{\mathbb{Z}}$ generated by $(x^i_r)_{i\in\mathbb{Z},\,r=0,\cdots,t}$, and $\underline{\underline{\mathcal{F}_t}}$ the σ-algebra over $\mathcal{S}^{\mathbb{Z}}$ generated by $(y^i_r)_{i\in\mathbb{Z},\,r=0,\cdots,t}$. We define the mapping $L:\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})\to\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ by

$$\mu\to L(\mu)=Q^\mu.$$

It follows from (9) that

$$Q^\mu_{|\mathcal{F}_0}=\mu_I^{\mathbb{Z}}, \tag{91}$$

which is independent of μ.

It may be inferred from the definitions in Section 4.1.2 that the marginal of $L(\mu)=Q^\mu$ over $\mathcal{F}_t$ only depends upon the marginal of μ over $\mathcal{F}_{t-1}$. This follows from the fact that $\underline{\underline{Q}}^\mu|_{\underline{\underline{\mathcal{F}_s}}}$ (which determines $Q^\mu|_{\mathcal{F}_s}$) is completely determined by the means $\{c^\mu_t; t=0,\ldots,s-1\}$ and covariances $\{K^{\mu,j}_{uv}; j\in\mathbb{Z}, u,v\in[0,s-1]\}$. In turn, it may be observed from (36) and (39) that these variables are determined by $\mu|_{\mathcal{F}_{s-1}}$. Thus for any $\mu,\nu\in\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}})$ and $t\in[1,T$, if

$$\mu_{|\mathcal{F}_{t-1}}=\nu_{|\mathcal{F}_{t-1}},$$

then

$$L(\mu)_{|\mathcal{F}_t}=L(\nu)_{|\mathcal{F}_t}.$$

It follows from repeated application of the above identity that for any ν satisfying $\nu_{|\mathcal{F}_0} = \mu_I^{\mathbb{Z}}$,

$$L^T(\nu)_{|\mathcal{F}_T} = L(L^T(\nu))_{|\mathcal{F}_T}. \tag{92}$$

Defining

$$\mu_e = L^T(\nu), \tag{93}$$

it follows from (92) that μ_e satisfies (90).

Conversely if $\mu = L(\mu)$ for some μ, then we have that $\mu = L^2(\nu)$ for any ν such that $\nu_{|\mathcal{F}_{T-2}} = \mu_{|\mathcal{F}_{T-2}}$. Continuing this reasoning, we find that $\mu = L^T(\nu)$ for any ν such that $\nu_{|\mathcal{F}_0} = \mu_{|\mathcal{F}_0}$. But by (91), since $Q^\mu = \mu$, we have $\mu_{|\mathcal{F}_0} = \mu_I^{\mathbb{Z}}$. But we have just seen that any μ satisfying $\mu = L^T(\nu)$, where $\nu_{|\mathcal{F}_0} = \mu_I^{\mathbb{Z}}$, is uniquely defined by (93), which means that $\mu = \mu_e$. ☐

Theorem 42. Π^N *converges weakly to* δ_{μ_e}, *i.e., for all* $\Phi \in \mathcal{C}_b(\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}}))$,

$$\lim_{N\to\infty} \int_{\mathcal{T}^N} \Phi(\hat{\mu}^N(x))\, Q^N(dx) = \Phi(\mu_e)$$

Proof. The proof follows directly from the existence of an LDP for the measure Π_N, see theorem 2, and is a straightforward adaptation of the one in [34, Theorem 2.5.1]. ☐

We may use the proof of proposition 41 to characterize the unique measure μ_e such that $\mu_e = Q^{\mu_e}$ in terms of its image $\underline{\underline{\mu_e}}$. This characterization allows one to directly numerically calculate μ_e. We characterize $\underline{\underline{\mu_e}}$ recursively (in time), by providing a method of determining $\underline{\underline{\mu_e}}_{|\underline{\underline{\mathcal{F}_t}}}$ in terms of $\underline{\underline{\mu_e}}_{|\underline{\underline{\mathcal{F}_{t-1}}}}$. However we must firstly outline explicitly the bijective correspondence between $\mu_{e|\mathcal{F}_t}$ and $\underline{\underline{\mu_e}}_{|\underline{\underline{\mathcal{F}_t}}}$, as follows. For $v \in \mathcal{S}$, we write $\Psi^{-1}(v) = (\Psi^{-1}(v)_0, \dots, \Psi^{-1}(v)_T)$. We recall from (9) that $\Psi^{-1}(v)_0 = \Psi_0^{-1}(v_0)$. The coordinate $\Psi^{-1}(v)_t$ is the affine function of v_s, $s = 0 \cdots t$ obtained from equations (11) and (12)

$$\Psi^{-1}(v)_t = \sum_{i=0}^{t-1} \gamma^i v_{t-i} + \gamma^t \Psi_0^{-1}(v_0) + \bar{\theta}\frac{\gamma^t - 1}{\gamma - 1}.$$

Let $\Psi_{(t)}^{-1}(v) : \mathbb{R}^{t+1} \to \mathbb{R}^{t+1}$ be such that

$$\Psi_{(t)}^{-1}(v_0, \dots, v_t) = (\Psi_0^{-1}(v_0), \Psi^{-1}(w)_1, \dots, \Psi^{-1}(w)_t).$$

where $w \in \mathcal{S}$ is such that $w_s = v_s$ for $0 \leq s \leq t$. With the same notations as in definition 3 we have $\mu|_{\mathcal{F}_t} = \underline{\underline{\mu}}|_{\underline{\underline{\mathcal{F}_t}}} \circ \Psi_{(t)} \circ f^{-1}$.

In the course of the previous proof we saw that $\mu_{e|\mathcal{F}_0} = \mu_I^{\otimes \mathbb{Z}}$ and $\underline{\underline{\mu_{e}}}_{|\underline{\underline{\mathcal{F}_0}}} = \mathcal{N}(0, \sigma^2)^{\otimes \mathbb{Z}}$, which gives us the first step in our induction. It remains for us to explicitly outline how we determine $\underline{\underline{\mu_{e}}}_{|\underline{\underline{\mathcal{F}_t}}}$ from $\underline{\underline{\mu_{e}}}_{|\underline{\underline{\mathcal{F}_{t-1}}}}$ for each $t \geq 1$. We saw in the previous proof that both of these are Gaussian Processes. As was explained, it suffices for us to provide expressions for $c_t^{\mu_e}$ and $\{K_{st}^{\mu_e, j}, s = 0, \ldots, t, j \in \mathbb{Z}\}$ in terms of $\underline{\underline{\mu_{e}}}_{|\underline{\underline{\mathcal{F}_{t-1}}}}$ (note that $K^{\mu_e, j}$ is symmetric). The other components of the mean and covariance of $\underline{\underline{\mu_{e}}}_{|\underline{\underline{\mathcal{F}_t}}}$ are the same as their analogues in $\underline{\underline{\mu}}_{|\underline{\underline{\mathcal{F}_{t-1}}}}$. The mean is given by

$$c_t^{\mu_e} = \bar{J} \int_{]0,1[^t} y_{t-1} \, \mu^1_{e|\mathcal{F}_{t-1}}(dy) = \bar{J} \int_{\mathbb{R}^t} \left(f \circ \Psi^{-1}_{(t-1)}(v)_{t-1} \right) \underline{\underline{\mu^1_{e}}}_{|\underline{\underline{\mathcal{F}_{t-1}}}}(dv),$$

where μ_e^1 is the marginal distribution over one neuron.

The formula for $K^{\mu_e, j}$ can be obtained from equations (39) and (37). Indeed, we have

$$K^{\mu_e, j} = \theta^2 \mathrm{OZ}_{T+1} \, {}^t\mathrm{OZ}_{T+1} + \sum_{l=-\infty}^{\infty} \Lambda(j, l) M^{\mu_e, l},$$

and

$$M_{rs}^{\mu_e, l} = \int_{\mathcal{T}^{\mathbb{Z}}} y^0_{r-1} y^l_{s-1} d\mu_e(y) \quad r, \, s \geq 1.$$

If $r = 0$ or $s = 0$, then $M_{rs}^{\mu_e, k} = 0$. This can be rewritten, in the case of $1 \leq r, s \leq t$, as

$$M_{rs}^{\mu_e, l} = \int_{\mathbb{R}^t \times \mathbb{R}^t} \left(f \circ \Psi^{-1}_{(t-1)}(v^0)_{r-1} \right) \times \left(f \circ \Psi^{-1}_{(t-1)}(v^l)_{s-1} \right) \underline{\underline{\mu_e}}^{(0,l)}{}_{|\underline{\underline{\mathcal{F}_{t-1}}}}(dv^0 dv^l).$$

Here $\underline{\underline{\mu_e}}^{(0,l)}{}_{|\underline{\underline{\mathcal{F}_{t-1}}}}(dv^0 dv^l)$ is distributed as $\mathcal{N}_{2t}((c^{\mu_e}_{(t-1)}, c^{\mu_e}_{(t-1)}), K^{\mu_e,(0,l)}_{(t-1)})$, where $c^{\mu_e}_{(t-1)} = (c_0^{\mu_e}, \ldots, c^{\mu_e}_{t-1})$,

$$K^{\mu_e,(0,l)}_{(t-1)} = \begin{bmatrix} K^{\mu_e,0}_{(t-1)} & K^{\mu_e,l}_{(t-1)} \\ K^{\mu_e,l}_{(t-1)} & K^{\mu_e,0}_{(t-1)} \end{bmatrix},$$

and $K^{\mu_e,l}_{(t-1)}$ is the $t \times t$ submatrix of $K^{\mu_e,l}$ composed of the elements from times 0 to $(t-1)$.

One cannot in practice numerically calculate all of the $K^{\mu_e,j}_{(t)}$ at each time step, as there are an infinite number of neurons. However since $\Lambda(j,k)$ must decay to zero as either j or k asymptotes to infinity, we strongly expect that if we only simulate N neurons, then the results will converge as $N \to \infty$. We note that numerical simulation using the above procedure would likely be highly unstable as one would expect errors to accumulate exponentially. It is possible that numerical simulation of the spectral densities would be much more accurate. We will explore these questions further in a subsequent paper.

7 Conclusion

In this section we sketch out some important consequences of our work and possible generalizations.

7.1 Important consequences

We note that the LDP of Moynot and Samuelides [34, 35] may be obtained from ours by stipulating that $\Lambda(a,b)$ is nonzero if and only if $a = b = 0$. Their LDP may then be obtained by applying a contraction principle to our LDP through taking the 1-dimensional marginal of $\hat{\mu}^N$. More generally, for any $d \in \mathbb{Z}^+$ one may obtain a process-level LDP governing the interaction of each neuron with its d neighbours by applying a contraction principle to the $d-$dimensional marginal of the empirical measure.

We state some implications of our results (particularly theorem 2).

Corollary 43. *For all $h \in \mathcal{C}_b(\mathcal{T}^{\mathbb{Z}})$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=-n}^{n} \int_{\mathcal{T}^N} h(S^i(x(N)))\, Q^N(dx) = \int_{\mathcal{T}^{\mathbb{Z}}} h(x)\, d\mu_e(x)$$

Proof. It is sufficient to apply theorem 42 in the case where Φ in $\mathcal{C}_b(\mathcal{M}^+_{1,s}(\mathcal{T}^{\mathbb{Z}}))$ is defined by

$$\Phi(\mu) = \int_{\mathcal{T}^{\mathbb{Z}}} h\, d\mu$$

□

Since the proof of theorem 42 only requires the use of the rightmost inequality in the definition of the LDP, we can in fact obtain a quenched convergence result through the use of the Borel-Cantelli lemma. We recall that $Q^N(J, \Theta)$ is the conditional law of N neurons for given J and Θ.

Theorem 44. *For each closed set F of $\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}})$ and for almost all (J, θ)*

$$\lim_{N\to\infty} \frac{1}{N} \log \left[Q^N(J,\theta)(\hat{\mu}^N \in F) \right] \leq - \inf_{\mu \in F} H(\mu).$$

Proof. The proof is a combination of Tchebyshev's inequality and the Borel-Cantelli lemma and is a straightforward adaptation of the one in [34, Theorem 2.5.4, Corollary 2.5.6]. □

This result allows us to state the quenched analog to theorem 42.

Corollary 45. *For all $\Phi \in \mathcal{C}_b(\mathcal{M}_{1,s}^+(\mathcal{T}^{\mathbb{Z}}))$ and for almost all (J, θ) we have*

$$\lim_{N\to\infty} \int_{\mathcal{T}^N} \Phi(\hat{\mu}^N(x))\, Q^N(J,\theta)(dx) = \Phi(\mu_e)$$

7.2 Possible extensions

Our results hold true if we assume that equation (1) is replaced by the more general equation

$$U_t^j = \sum_{k=1}^{l} \gamma_k U_{t-k}^j + \sum_{i=-n}^{n} J_{ji} f(U_{t-1}^i) + \theta_j + B_{t-1}^j, \quad j = -n, \ldots, n \quad t = l, \ldots, T,$$

where l is a positive integer strictly less than T (in practice much smaller). This equation accounts for a more complicated "intrinsic" dynamics of the neurons, i.e. when they are uncoupled. The parameters γ_k, $k = 1 \cdots l$ must satisfy some conditions to ensure stability of the uncoupled dynamics.

The extension to continuous time is certainly worth considering even though we expect it to be quite difficult.

The hypothesis that the synaptic weights are Gaussian is somewhat unrealistic from the biological viewpoint. In his PhD thesis [34], Moynot has obtained some promising preliminary results in the case of uncorrelated weights. We think that this is also a promising avenue.

Moynot again, in his thesis, has extended the uncorrelated weights case, to include two populations with different (Gaussian) statistics for each population. This is also an important practical problem in neuroscience. Extending Moynot's result to the correlated case is probably a low hanging fruit.

Last but not least, the solutions of the equations for the mean and covariance operator of the measure minimizing the rate function derived in section 6 and their numerical simulation are very much worth investigating and their predictions confronted to biological measurements.

7.3 Discussion

In recent years there has been a lot of effort to mathematically justify neural-field models, through some sort of asymptotic analysis of finite-size neural networks. Many, if not most, of these models assume / prove some sort of thermodynamic limit, whereby if one isolates a particular population of neurons in a localised area of space, they are found to fire increasingly asynchronously as the number in the population asymptotes to infinity.[6] Indeed this was the result of Moynot and Samuelides. However our results imply that there are system-wide correlations between the neurons, even in the asymptotic limit. The key reason why we do not have propagation of chaos is that the Radon-Nikodym derivative $\frac{dQ^N}{dP^N}$ of the average laws in proposition 3 cannot be tensored into N i.i.d. processes; whereas the simpler assumptions on the weight function Λ in Moynot and Samuelides allow the Radon-Nikodym derivative to be tensored. A very important implication of our result is that the mean-field behaviour is insufficient to characterise the behaviour of a population. Our limit process μ_e is system-wide and ergodic. Our work challenges the assumption held by some that one cannot have a 'concise' macroscopic description of a neural network without an assumption of asynchronicity at the local population level.

The utility of this paper extends well beyond the identification of the limit law μ_e. The LDP provides a powerful means of assessing how quickly the empirical measure converges to its limit. In particular, it provides a means of assessing the probability of finite size effects. For example if it could be shown that the rate function H is sharply convex everywhere, then one would be more confident that the system converges quickly to its limit law. The rate functions of many classical LDPs, such as the one in lemma 32, are

[6] We noted in the introduction that this is termed propagation of chaos by some.

indeed convex (in fact the rate function $H^\nu(\cdot)$, for fixed ν, is affine). However it is not clear whether our rate function H is convex. Indeed if it could be shown that the rate function H is not sharply convex, and in particular that it has a local minimum at another point μ_m, then perhaps if N is not too great there could be a reasonable probability that the empirical measure lies close to μ_m. The upshot of this discussion is that further exploration of the topology of the rate function H could be a very fruitful avenue of research for assessing the probability of finite-size effects. It would be of interest to compare our LDP with other analyses of the rate of convergence of neural networks to their limits as the size asymptotes to infinity. This includes the system-size expansion of Bressloff [5], the path-integral formulation of Buice and Cowan [6] and the systematic expansion of the moments by (amongst others) [28, 22, 7].

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A A comment on lemma 32

We firstly suppose that $c^\nu = 0$, so that $\underline{\underline{Q}}^\nu$ is a centred stationary Gaussian Process. We denote the corresponding image law by $\underline{\underline{\Pi}}_0^{\nu,N}$. There exists an LDP for $\underline{\underline{\Pi}}_0^{\nu,N}$ with good rate function

$$\underline{\underline{H}}_0^\nu(\underline{\underline{\mu}}) = I^{(3)}(\underline{\underline{\mu}}, \underline{\underline{P}}^{\mathbb{Z}}) - \frac{1}{2\sigma^2}\mathbb{E}^{\underline{\underline{\mu}}}(\|v^0\|^2) - (T+1)\log\sigma + \\ \frac{1}{4\pi\sigma^2}\int_{-\pi}^{\pi}(\mathrm{Id}_{T+1} - \tilde{A}^\nu(\omega)) : \tilde{v}^\mu(d\omega) + \frac{1}{4\pi}\int_{-\pi}^{\pi}\log\det(\sigma^2\mathrm{Id}_{T+1} + \tilde{K}^\nu(\omega))\,d\omega. \tag{94}$$

If $\mu \notin \mathcal{E}_2$, then the spectral density $\tilde{v}^\mu$ does not exist (as we noted in (30)) and H_0^μ is infinite. We now comment on how this expression (94) is obtained.

The existence of an LDP for $\underline{\underline{\Pi}}_0^{\nu,N}$ was proved by Baxter and Jain [2], although they did not provide an explicit expression for the rate function. The above expression for the rate function may be obtained through a straightforward extension of the proof in Donsker and Varadhan [21]. Donsker and Varadhan proved their expression (labelled (1.9) in their paper) in the case of $\mathbb{R}^{\mathbb{Z}}$, finding (in their notation)

$$H_f(R) = \mathbb{E}^R\left[\int_{-\infty}^{\infty} r(y|\omega)\log r(y|\omega)dy\right] + \frac{1}{2}\log 2\pi \\ + \frac{1}{4\pi}\int_0^{2\pi}\frac{dG(\gamma)}{f(\gamma)}d\gamma + \frac{1}{4\pi}\int_0^{2\pi}\log f(\gamma)d\gamma. \quad (95)$$

Here R is a stationary measure in $\mathcal{M}_{1,s}^+(\mathbb{R}^{\mathbb{Z}})$ with regular conditional probability distribution $r(y|\omega)$ and continuous spectral density $G(\gamma)$. The stationary Gaussian process against which the entropy is taken has continuous spectral density $f(\gamma) : [0, 2\pi] \to \mathbb{R}$ and zero mean. We briefly explain how this expression corresponds to ours.

Donsker and Varadhan obtained their expression $H_f(R)$ through a similar technique to ours: they 'diagonalise' the covariance operator using a 'moving average' transformation (note that the diagonalised variables in Donsker and Varadhan have variance 1, whereas in our case they have variance σ^2). It is easily shown that the transformed operator (which is analogous to our $\underline{\underline{P}}^{\mathbb{Z}}$) satisfies an LDP because the variables are independent and identically distributed. Recall that, in our model, the entropy of the 'diagonalised' system is $I^{(3)}(\mu, P^{\mathbb{Z}}) = \lim_{N\to\infty} N^{-1} I^{(2)}(\mu^N, P^{\otimes N})$. In fact the terms $I^{(3)}(\underline{\underline{\mu}}, \underline{\underline{P}}^{\mathbb{Z}}) - \frac{1}{2\sigma^2}\mathbb{E}^{\underline{\underline{\mu}}}[\|v^0\|^2]$ in (94) correspond to the terms $\mathbb{E}^R[\int_{-\infty}^{\infty} r(y|\omega)\log r(y|\omega)dy] + \frac{1}{2}\log 2\pi$ in (95) (this may be inferred from the expression (99) for $I^{(2)}(\mu^N, P^{\otimes N})$).

The expression

$$\frac{1}{4\pi\sigma^2}\int_{-\pi}^{\pi}(\mathrm{Id}_{T+1} - \tilde{A}^\nu(\omega)) : \tilde{v}^\mu(d\omega) \\ = \lim_{N\to\infty} N^{-1}\mathbb{E}^{\underline{\underline{\mu}}^N}\left[\frac{1}{2}{}^t v\left(\mathrm{Id}_{N(T+1)} - A^{\nu,N}\right)v\right]$$

may be thought of as the asymptotic limit of the expectation of the quadratic form induced by the inverse covariance operator. It corresponds to $\frac{1}{4\pi}\int_0^{2\pi}\frac{dG(\gamma)}{f(\gamma)}$

in (95). Finally, the terms in (94) of the form

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\det(\sigma^2\mathrm{Id}_{(T+1)}+\tilde{K}^{\nu}(\omega))\,d\omega-(T+1)\log(\sigma)$$
$$=\lim_{N\to\infty}\frac{1}{2}N^{-1}\log\left(\frac{\det\left(\sigma^2\mathrm{Id}_{N(T+1)}+K^{\nu,N}\right)}{\sigma^{2(T+1)N}}\right)=\Gamma_1(\nu),$$

are the asymptotic limit of the logarithm of the ratio of the determinant of the original system divided by the determinant of the 'diagonalised' system. They corresponds to the term $\frac{1}{4\pi}\int_0^{2\pi}\log f(\gamma)d\gamma$ in (95). The extension of the proof in Donsker and Varadhan to our case is straightforward because $\mathcal{S}=\mathbb{R}^{T+1}$ is finite-dimensional and $N^{-1}\log\det\left(\sigma^2\mathrm{Id}_{N(T+1)}+K^{\nu,N}\right)$ is bounded below by $2\log\sigma$, although we must omit the details because of a shortage of space.

We now use the rate function (94) governing the zero-mean process to establish an LDP for a process with nonzero mean c^{ν}. Let $\Theta: M_{1,s}^{+}(\mathcal{S}^{\mathbb{Z}})\to M_{1,s}^{+}(\mathcal{S}^{\mathbb{Z}})$ be the translation map, such that for measureable A, $\Theta(\mu)(A)=\mu(A+c^{\nu})$. Since

$$\mathbb{E}^{\mu}[v^{0t}v^{k}]=\mathbb{E}^{\Theta(\mu)}[\left(v^0-c^{\nu}\right){}^t\left(v^k-c^{\nu}\right)],\tag{96}$$

we find that

$$\tilde{v}^{\mu}(d\omega)=\tilde{v}^{\Theta(\mu)}(d\omega)+2\pi\delta(\omega)\left(c^{\nu}\;{}^tc^{\nu}-c^{\nu}\mathbb{E}^{\Theta(\mu)}[{}^tv^0]-\mathbb{E}^{\Theta(\mu)}[v^0]\;{}^tc^{\nu}\right).\tag{97}$$

Since $\underline{\underline{\hat{\mu}}}^N(v^{-n},\cdots,v^n)$ is the image of $\underline{\underline{\hat{\mu}}}^N(v^{-n}-c^{\nu},\cdots,v^n-c^{\nu})$ under Θ, it follows from the Contraction Principle that there exists an LDP for $\underline{\underline{\Pi}}^{\nu,N}$ with rate function $\underline{\underline{H}}^{\nu}(\underline{\underline{\mu}}):=\underline{\underline{H}}_0^{\nu}(\Theta^{-1}(\underline{\underline{\mu}}))$. Since $\tilde{A}^{\nu}(0)$ is symmetric, we may infer from (94) and (97) that

$$\underline{\underline{H}}^{\nu}(\underline{\underline{\mu}})=I^{(3)}(\Theta^{-1}(\underline{\underline{\mu}}),\underline{\underline{P}}^{\mathbb{Z}})-\frac{1}{4\pi\sigma^2}\int_{-\pi}^{\pi}\tilde{A}^{\nu}(\omega):\tilde{v}^{\mu}(d\omega)$$
$$-\frac{1}{2\sigma^2}\left({}^tc^{\nu}\tilde{A}^{\nu}(0)c^{\nu}-2\;{}^tc^{\nu}\tilde{A}^{\nu}(0)\mathbb{E}^{\underline{\underline{\mu}}}[v^0]\right)+\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\det(\mathrm{Id}_{T+1}+\sigma^{-2}\tilde{K}^{\nu}(\omega))\,d\omega.\tag{98}$$

We now determine an explicit expression for $I^{(3)}(\Theta^{-1}(\underline{\underline{\mu}}),\underline{\underline{P}}^{\mathbb{Z}})$. If $\underline{\underline{\mu}}^N$ does not possess a density then $I^{(2)}(\underline{\underline{\mu}}^N,\underline{\underline{P}}^{\otimes N})$ is infinite (because $\underline{\underline{P}}^{\otimes N}$ possesses a

density, and therefore $\underline{\underline{\mu}}^N$ is not absolutely continuous with respect to $\underline{\underline{P}}^{\otimes N}$). Otherwise, let the density of $\underline{\underline{\mu}}^N$ be $r^{\mu^N}(v^{-n}, \ldots, v^n)$. We find that

$$I^{(2)}(\underline{\underline{\mu}}^N, \underline{\underline{P}}^{\otimes N}) = \int_{\mathcal{S}^N} \log\left(\frac{r^{\underline{\underline{\mu}}^N}(v)}{(2\pi\sigma^2)^{-\frac{N(T+1)}{2}} \exp(-\frac{1}{2\sigma^2}\|v\|^2)} \right) r^{\mu^N}(v) dv, \quad (99)$$

where $dv = \prod_{j=-n}^{n} \prod_{s=0}^{T} dv_s^j$. Upon expansion, we find that

$$\begin{aligned} I^{(2)}(\Theta^{-1}(\underline{\underline{\mu}}^N), \underline{\underline{P}}^{\otimes N}) = \int_{\mathcal{S}^N} &\left(\log\left(r^{\mu^N}(v + c^\nu) \right) \right. \\ &\left. + \frac{N(T+1)}{2} \log(2\pi\sigma^2) + \frac{1}{2\sigma^2}\|v\|^2 \right) r^{\mu^N}(v + c^\nu) dv. \end{aligned}$$

Thus

$$I^{(2)}(\Theta^{-1}(\underline{\underline{\mu}}^N), \underline{\underline{P}}^{\otimes N}) = I^{(2)}(\underline{\underline{\mu}}^N, \underline{\underline{P}}^{\otimes N}) - \frac{N}{\sigma^2}\langle c^\nu, \mathbb{E}^{\underline{\underline{\mu}}}[v^0]\rangle - \frac{N}{2\sigma^2}\|c^\nu\|^2. \quad (100)$$

Noting that $\mathbb{E}^{\underline{\underline{\mu}}}[v^0] = \bar{v}^\mu$ (definition 4), the identity (77) now follows from (63), (72), (73), (98) and (100).

B Proof of Proposition 37

Proof. We have already proved that Γ_1 is continuous in proposition 19. It thus suffices for us to prove that for some constant C_I,

$$|\Gamma_2^\nu(\mu) - \Gamma_2(\mu)| \leq C_I \varepsilon \left(1 + I^{(3)}(\mu, P^{\mathbb{Z}}) \right). \quad (101)$$

Fix ϵ. We define $\Phi^N(x)$ to be the integrand of $CN(\Gamma_2(\mu^N) - \Gamma_2^\nu(\mu^N))$ for some positive constant C. Taking the expression from the proof of lemma 35, we have

$$\Phi^N = \frac{C}{2\sigma^2}\left(\frac{1}{N}\sum_{l=-n}^{n} {}^t y^l \left(\tilde{A}^{\mu^N,l} - \tilde{A}^{\nu,l} \right) y^l + 2\langle d_{\nu,\mu}, y^0\rangle + N e_{\nu,\mu} \right).$$

Here y is defined in (81). This is a continuous real function on $\mathcal{T}^N$ which is unbounded. We have that

$$\Phi^N \leq \frac{C}{2\sigma^2}\left(\frac{1}{N}\sum_{l=-n}^{n} \|\tilde{A}^{\mu^N,l} - \tilde{A}^{\nu,l}\| \|y^l\|^2 + 2\langle d_{\nu,\mu}, y^0\rangle + N|e_{\nu,\mu}| \right).$$

In turn, using the identities in lemma 35 we find that $\Phi^N \leq \Phi^N_{max}$ where

$$\Phi^N_{max} = \frac{C}{2\sigma^2}\left(\frac{1}{N}C^*(C^\nu_N+\epsilon)\sum_{l=-n}^{n}\|y^l\|^2 + 2\langle d_{\nu,\mu}, y^0\rangle + NC^*(C^\nu_N+\epsilon)\right).$$

Note that Φ^N_{max} is integrable with respect to μ because $\mu \in \mathcal{E}_2$. For $M > 0$, let

$$B^{\nu,\mu}_{N,M} = \{y : \Phi^N_{max}(y) \leq 0 \text{ or } \|y\|^2 \leq NM\}.$$

It may be observed that $B^{\nu,\mu}_{N,M}$ is compact, because the eigenvalues of the quadratic form in Φ^N_{max} are strictly positive. It follows that Φ^N and Φ^N_{max} are bounded over $B^{\nu,\mu}_{N,M}$ (for all M). Let Φ^N_M be Φ^N multiplied by the indicator function over the set $B^{\nu,\mu}_{N,M}$. Since Φ^N_M is bounded and continuous, we find from the Fenchel-Legendre transform that

$$\int_{\mathcal{T}^N} \Phi^N_M(y)\,\mu^N(dx) \leq \log\int_{\mathcal{T}^N} \exp\Phi^N_M(y)P^{\otimes N}(dx) + I^{(2)}(\mu^N, P^{\otimes N}). \quad (102)$$

In order that $\int_{\mathcal{T}^N}\exp\Phi^N_{max}(y)P^{\otimes N}(dx) < \infty$, we stipulate that

$$C = \frac{C_1}{C^*(C^\nu_N+\varepsilon)},$$

for some constant $0 < C_1 < 1$, where C^* is given in lemma 35. Since $\Phi^N_M \leq \Phi^N_{max}$ over $B^{\nu,\mu}_{N,M}$, we may apply the dominated convergence theorem to (102) (taking $M \to \infty$) to obtain

$$\int_{\mathcal{T}^N} \Phi^N(y)\,\mu^N(dx) \leq \log\int_{\mathcal{T}^N} \exp\Phi^N_{max}(y)\,P^{\otimes N}(dx) + I^{(2)}(\mu^N, P^{\otimes N}).$$

We observe from proposition 1 and equation (82) that, under the transformation $h : v \to y$, $\underline{\underline{P}}^{\otimes N}(dv)$ becomes

$$\underline{\underline{P}}^{\otimes N}\circ h^{-1}(dy) = \bigotimes_{l=-n}^{n}\mathcal{N}(\mathbf{0}_{T+1}, N\sigma^2\mathrm{Id}_{T+1})dy^l.$$

An application of lemma 6 thus yields

$$\int_{\mathcal{T}^N}\exp\Phi^N_{max}(y)P^{\otimes N}(dx) = \exp\frac{NC_1}{2\sigma^2}\times$$
$$(1-C_1)^{-\frac{N(T+1)}{2}} \times \exp\frac{NC^2}{2\sigma^2(1-C_1)}\|d_{\nu,\mu}\|^2. \quad (103)$$

We use the fact (proved in lemma 35) that $\|d_{\nu,\mu}\| \leq C^*(C_N^\nu + \varepsilon)$ to find that

$$NC(\Gamma_2(\mu^N) - \Gamma_2^\nu(\mu^N)) \leq Ns + I^{(2)}(\mu^N, P^{\otimes N})$$

where

$$s = \frac{C_1}{2\sigma^2} - \frac{T+1}{2}\log(1 - C_1) + \frac{C_1^2}{2\sigma^2(1 - C_1)}.$$

We divide by NC and take the limit as $N \to \infty$. The result (101) follows since, by lemma 34, $C_N^\nu \to 0$ as $N \to \infty$. □

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